Target close execution strategies*

Olivier Guéant[†]

Abstract

Be it directly, using DMA services, or intermediated by brokers, investors send their orders to trading platforms using execution algorithms. VWAP algorithms are widely used but they are only part of the menu proposed to investors. POV (also called PVol, for Percentage of Volume) is another instance of execution strategy, in which participation rate to the market is as close as possible to a predetermined constant. IS (Implementation Shortfall) or TWAP (Time weighted average price) algorithms are also proposed to minimize slippage with respect to the arrival price of the order or to the average price over the desired execution period. Numerous investors also use another execution strategy in line with the evaluation of their portfolio: Target Close strategy. When their portfolio is evaluated every day at the closing price of the day, investors' interest is to be executed as close as possible to the closing price. In this article dedicated to Target Close strategies, we develop several liquidation strategies to execute as close as possible to the closing price, without making the closing price. Risk-liquidity premia are also discussed.

Introduction

Investors in stocks buy and sell large quantities of shares to build or rebalance their portfolios. Depending on the urgency of their orders and on their incentives, they use different execution algorithms. Algorithms trying to replicate the VWAP over a predetermined time window are the most widely used in practice although, only a few academic papers are dedicated to it (see [12, 13, 14, 16, 23, 28, 29]). IS strategies, rarely used in comparison, have been the topic of most academic works on optimal execution. The most classical framework is the one developed by Almgren and Chriss in their seminal papers [7, 8, 10]. This framework, enriched to account for stochastic volatility and liquidity [6], and generalized to other objective criteria and to more general impact functions (see [9], [15], [18], [27], [32], [33] or [34]) has long been the only reference framework. Today, this framework is challenged by new

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[†]Université Paris-Diderot, UFR de Mathématiques, Laboratoire Jacques-Louis Lions. Avenue de France, 75013 Paris, France. olivier.gueant@ann.jussieu.fr

models involving transient market impact ([1], [2], [3], [30] and [31]), and completed by new papers focused on the tactical layer, that is on the actual way to proceed, using for instance dark pools [24, 25, 26] or limit orders [11, 19, 20].
If VWAP[1] and IS strategies have been studied, POV strategies have long been ignored by academics and we only know our study [21] on the subject. When it comes to Target Close strategies, no academic paper is available. This paper is aimed at filling this blank. Target Close strategies are used by investors in order to get a price as close as possible to the closing price. Their incentive is most of the time that their portfolio is evaluated mark-to-market, using closing prices.
In this paper, we will successively consider a market without and with a closing auction. When there is no closing auction, Target Close strategies use the continuous auction to execute orders. The trade-off between execution costs and price risk is the same as for IS orders, although the benchmark price is the price at the end of the period and not the arrival price. The problem is more complicated in presence of a closing auction. Obtaining the closing price is then possible but no one wants to impact the closing price so as to make it. Hence, for large orders, a part of it must be executed during the continuous auction, before the closing auction. We propose in this paper two models. In the first model, the benchmark is still the closing price but we impose an upper bound to the volume obtained at the closing auction. In the second model, the benchmark price is a convex combination of the closing price and of the price at the end of the continuous auction.

In Section 1, we introduce a model without closing auction and we provide a result linking IS strategies and Target Close strategies in this framework. We then provide an efficient way to compute Target Close trading curves. In Section 2, we add a closing auction and we provide several liquidation models that use both the continuous auction and the closing auction.

1 The model without closing auction

1.1 Setup of the model

Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ satisfying the usual conditions. We assume that all stochastic processes are defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$.

We consider an investor with a portfolio of stocks. This investor is willing to change his exposition to a particular stock and he wants to have sold at time T (supposed to be the end of day) q_0 shares of the specified stock.[2] The velocity at which trading takes place depends on market conditions. In particular, we introduce a process $(V_t)_{t\in[0,T]}$ for the market volume. This process is assumed to be continuous, deter-

[1] TWAP strategies can be seen as special cases of VWAP.
[2] If q_0 is negative, then it corresponds to a buy order.

ministic,[3] and such that $\exists \underline{V} > 0, \overline{V} > 0, \forall t \in [0,T], \underline{V} \le V_t \le \overline{V}$.

To model the execution process, we introduce an inventory process $(q_t)_{t\in[0,T]}$ defined by:

$$\forall t \in [0,T], \quad q_t = q_0 - \int_0^t v_s ds,$$

where the strategy $(v_s)_{s\in\mathbb{R}_+}$ belongs to the admissible set

$$\mathcal{A} = \left\{ (v_t)_{t\in[0,T]}, \text{ progressively measurable}, v \in L^\infty(\Omega \times [0,T]), \int_0^T v_s ds = q_0 \text{ a.s.} \right\}.$$

As in the classical Almgren-Chriss framework [7, 8, 10] (see also [18, 33]), we consider that trades impact market prices in two distinct ways. Firstly, there is a permanent market impact that imposes a drift to the price process $(S_t)_{t\in\mathbb{R}_+}$. As in [22], we consider a positive function f in $L^1(0, |q_0|)$ and we assume that:

$$dS_t = \sigma dW_t + f(|q_0 - q_t|)v_t dt, \qquad \sigma > 0, k \ge 0.$$

Secondly, the price obtained by the investor at time t is not S_t because of execution costs (or instantaneous market impact). To model this, we introduce a function $L \in C^1(\mathbb{R}_+, \mathbb{R})$ verifying the following hypotheses:[4]

- $L(0) = 0$,
- L is increasing,
- L is strictly convex,
- $\lim_{\rho\to+\infty} \frac{L(\rho)}{\rho} = +\infty$.

This allows to define the cash process $(X_t)_{t\in\mathbb{R}_+}$ as:

$$X_t = \int_0^t \left(v_s S_s - V_s L\left(\frac{v_s}{V_s}\right) - \psi |v_s| \right) ds,$$

where the execution cost is divided into two parts: a linear part that represents a fixed cost ($\psi \ge 0$) per share – linked to the bid-ask spread –, and a strictly convex part modeled by L.

In the case of an IS strategy (as in the usual Almgren-Chriss case), the goal is to minimize a risk-adjusted cost of execution where the benchmark is the initial mark-to-market value of the quantity to be sold: $q_0 S_0$. In other words, if we consider an expected utility framework, the usual Almgren-Chriss problem consists in maximizing over $v \in \mathcal{A}$ the objective function $\mathbb{E}\left[-\exp(-\gamma(X_T - q_0 S_0))\right] =$

[3]This assumption may seem odd. Practitioners usually consider market volume curves determined statistically to account for the intraday seasonality of market volume.

[4]We want to cover the cases $L(\rho) = \eta\rho^{1+\phi}$ for $\eta > 0$ and $\phi > 0$.

$-e^{\gamma q_0 S_0}\mathbb{E}\left[\exp(-\gamma X_T)\right]$. In the case of a Target Close strategy, the benchmark is not anymore $q_0 S_0$, but rather $q_0 S_T$. In other words, the central problem of this first section is to maximize over $v \in \mathcal{A}$ the objective function:

$$J(v) = \mathbb{E}\left[-\exp(-\gamma(X_T - q_0 S_T))\right],$$

where $\gamma > 0$ is the absolute risk aversion parameter of the investor.

An interpretation of the objective function is the following. If the investor wants to obtain the closing price S_T, then he can contract with an intermediary. The intermediary gets the shares at time 0, liquidate them over $[0, T]$, obtain X_T at time T and pays the investor $q_0 S_T$. Therefore, the intermediary must optimize the expected utility of $X_T - q_0 S_T$. An intermediary proposing such guaranteed closing price must price the service. Using indifference pricing, one can define the reserve price (or risk-liquidity premium) for the service as:

$$\ell_{TC}(q_0) = \frac{1}{\gamma}\log\left(-\sup_{v\in\mathcal{A}}\mathbb{E}\left[-\exp(-\gamma(X_T - q_0 S_T))\right]\right)$$

1.2 A correspondence between Target Close strategies and IS strategies

To start solving the problem, let us define the set $\mathcal{A}_{\text{det}}$ of deterministic strategies in $\mathcal{A}$. We shall show that there exists an optimal strategy and that this strategy can be searched among deterministic ones. The first step is to write the value of $X_T - q_0 S_T$:

Proposition 1. *Let us define $\Phi(z) = \int_0^z y f(|y|)dy$. Let us consider $v \in \mathcal{A}$. We have:*

$$X_T - q_0 S_T = \Phi(q_0) - \int_0^T V_s L\left(\frac{v_s}{V_s}\right) ds - \psi \int_0^T |v_s| ds + \int_0^T \sigma(q_s - q_0) dW_s.$$

In particular, if $v \in \mathcal{A}_{det}$, then:

$$X_T - q_0 S_T \sim \mathcal{N}\left(\Phi(q_0) - \int_0^T V_s L\left(\frac{v_s}{V_s}\right) ds - \psi \int_0^T |v_s| ds, \sigma^2 \int_0^T (q_s - q_0)^2 ds\right).$$

Proof:

By definition:

$$\begin{aligned}
X_T - q_0 S_T &= \int_0^T v_s S_s ds - q_0 S_T - \int_0^T V_s L\left(\frac{v_s}{V_s}\right) ds - \psi \int_0^T |v_s| ds \\
&= [(q_0 - q_t)S_t]_0^T - q_0 S_T + \int_0^T v_s f(|q_0 - q_s|)(q_0 - q_s) ds + \int_0^T \sigma(q_s - q_0) dW_s \\
&\quad - \int_0^T V_s L\left(\frac{v_s}{V_s}\right) ds - \psi \int_0^T |v_s| ds
\end{aligned}$$

$$= \quad \Phi(q_0) - \int_0^T V_s L\left(\frac{v_s}{V_s}\right) ds - \psi \int_0^T |v_s| ds + \int_0^T \sigma(q_s - q_0) dW_s.$$

If $(q_s)_{s\in[0,T]}$ is deterministic, then $X_T - q_0 S_T$ is normally distributed with mean $\Phi(q_0) - \int_0^T V_s L\left(\frac{v_s}{V_s}\right) ds - \psi \int_0^T |v_s| ds$ and variance $\sigma^2 \int_0^T (q_s - q_0)^2 ds$. □

Using the above Proposition and the Laplace transform of a normal distribution, we have straightforwardly a closed-form expression for the objective function J:

Corollary 1. *Let us consider $v \in \mathcal{A}_{det}$. Then:*

$$J(v) = -\exp\left(-\gamma\left(\Phi(q_0) - \int_0^T V_s L\left(\frac{v_s}{V_s}\right) ds \right.\right.$$

$$\left.\left. -\psi \int_0^T |v_s| ds - \frac{\gamma}{2}\sigma^2 \int_0^T (q_s - q_0)^2 ds\right)\right).$$

We can then define a new objective function for $v \in \mathcal{A}_{\text{det}}$ by

$$I(v) = \int_0^T V_s L\left(\frac{v_s}{V_s}\right) ds + \psi \int_0^t |v_s| ds + \frac{1}{2}\gamma\sigma^2 \int_0^T (q_s - q_0)^2 ds,$$

so that:

$$J(v) = -\exp\left(-\gamma\left(\Phi(q_0) - I(v)\right)\right),$$

and the maximizers of J correspond to the minimizers of I.

We introduce now the function

$$\begin{array}{rcccl} \mathcal{I}^{TC} & : & AC_{q_0,0}(0,T) & \to & \mathbb{R}_+ \\ & & q & \mapsto & \int_0^T \left(V_s L\left(\frac{q'(s)}{V_s}\right) + \psi|q'(s)| + \frac{1}{2}\gamma\sigma^2(q_0 - q(s))^2\right) ds, \end{array}$$

where $AC_{q_0,0}(0,T)$ is the set of absolutely continuous functions q on $[0,T]$ with $q(0) = q_0$ and $q(T) = 0$.

We shall relate this function to the objective function in the case of an IS strategy. This is the purpose of the following Proposition that is the basis of the correspondence theorem between IS strategies and Target Close strategies:

Proposition 2. *Let us define $\widehat{V} : t \in [0,T] \mapsto V_{T-t}$. Then q^* is a minimizer of $\mathcal{I}^{TC}$ if and only if $\tilde{q}^* : t \in [0,T] \mapsto q_0 - q^*(T-t)$ is a minimizer of:*

$$\begin{array}{rcccl} \mathcal{I}^{\tilde{IS}} & : & AC_{q_0,0}(0,T) & \to & \mathbb{R}_+ \\ & & \tilde{q} & \mapsto & \int_0^T \left(\widehat{V}_s L\left(\frac{\tilde{q}'(s)}{\widehat{V}_s}\right) + \psi|\tilde{q}'(s)| + \frac{1}{2}\gamma\sigma^2\tilde{q}(s)^2\right) ds, \end{array}$$

Proof:

For any $q \in AC_{q_0,0}(0,T)$, let us denote $\tilde{q} : t \in [0,T] \mapsto q_0 - q(T-t)$.

The proof of our result is based on the following identity:

$$\begin{aligned}
\mathcal{I}^{TC}(q) &= \int_0^T \left(V_s L\left(\frac{q'(s)}{V_s}\right) + \psi|q'(s)| + \frac{1}{2}\gamma\sigma^2(q_0 - q(s))^2 \right) ds \\
&= \int_0^T \left(V_s L\left(\frac{\tilde{q}'(T-s)}{V_s}\right) + \psi|\tilde{q}'(T-s)| + \frac{1}{2}\gamma\sigma^2\tilde{q}(T-s)^2 \right) ds \\
&= \int_0^T \left(V_{T-s} L\left(\frac{\tilde{q}'(s)}{V_{T-s}}\right) + \psi|\tilde{q}'(s)| + \frac{1}{2}\gamma\sigma^2\tilde{q}(s)^2 \right) ds \\
&= \int_0^T \left(\widehat{V}_s L\left(\frac{\tilde{q}'(s)}{\widehat{V}_s}\right) + \psi|\tilde{q}'(s)| + \frac{1}{2}\gamma\sigma^2\tilde{q}(s)^2 \right) ds \\
&= \mathcal{I}^{\tilde{IS}}(\tilde{q}).
\end{aligned}$$

Hence, the $\tilde{\cdot}$ operator defines a bijective correspondence between the minimizers of $\mathcal{I}^{TC}$ and the minimizers of $\mathcal{I}^{\tilde{IS}}$. This is the result of the Proposition. $\square$

This Proposition, along with the results of [18], allows to write the following theorem:

Theorem 1. *Let us define H the Legendre transform of L.*
There exists a unique minimizer $q^* \in AC_{q_0,0}(0,T)$ of the function $\mathcal{I}^{TC}$. It is a monotone function, independent of ψ, characterized by (E):

$$\left\{ \begin{array}{rcl} p'(t) &=& \gamma\sigma^2(q^*(t) - q_0) \\ {q^*}'(t) &=& V_t H'(p(t)) \end{array} \right. \qquad q^*(0) = q_0, \quad q^*(T) = 0.$$

Proof:

From [18], there exists a unique minimizer $\tilde{q}^* \in AC_{q_0,0}(0,T)$ of the function $\mathcal{I}^{\tilde{IS}}$. This minimizer is a monotone function, independent of ψ, characterized by $(\tilde{E})$:

$$\left\{ \begin{array}{rcl} \frac{d}{dt}P(t) &=& \gamma\sigma^2\tilde{q}^*(t) \\ \frac{d}{dt}\tilde{q}^*(t) &=& V_{T-t} H'(P(t)) \end{array} \right. \qquad \tilde{q}^*(0) = q_0, \quad \tilde{q}^*(T) = 0.$$

From Proposition 2, we then know that $q^* : t \in [0,T] \mapsto q_0 - \tilde{q}^*(T-t)$ is the unique minimizer of $\mathcal{I}^{TC}$, independent of ψ, characterized by:

$$\left\{ \begin{array}{rcl} P'(t) &=& \gamma\sigma^2(q_0 - q^*(T-t)) \\ {q^*}'(T-t) &=& V_{T-t} H'(P(t)) \end{array} \right. \qquad q^*(0) = q_0, \quad q^*(T) = 0.$$

Hence, denoting $p(t) = P(T-t)$, we have:

$$\left\{\begin{array}{rcl} -p'(T-t) & = & \gamma\sigma^2(q_0 - q^*(T-t)) \\ q^{*\prime}(T-t) & = & V_{T-t}H'(p(T-t)) \end{array}\right. \qquad q^*(0) = q_0, \quad q^*(T) = 0,$$

or equivalently the system

$$\left\{\begin{array}{rcl} p'(t) & = & \gamma\sigma^2(q^*(t) - q_0) \\ q^{*\prime}(t) & = & V_t H'(p(t)) \end{array}\right. \qquad q^*(0) = q_0, \quad q^*(T) = 0.$$

Reciprocally, if q is solution of (E), then using the same calculations as above, $\tilde{q} : t \in [0,T] \mapsto q_0 - q(T-t)$, is solution of $(\tilde{E})$ so that $\tilde{q}$ is the unique minimizer of $\mathcal{I}^{\tilde{IS}}$ (*i.e.* $\tilde{q} = \tilde{q}^*$) and eventually $q = q^*$. $\square$

As an example, we consider here the case where 120000 shares of a stock are to be liquidated by the end of the day using a Target Close strategy. The characteristics of the stock are the following: $\sigma = 0.6$, $\eta = 0.08$, $\phi = 0.5$ and $V_t = V = 1200000$. We assume that $\gamma = 6 \times 10^{-6}$. The resulting trading curve is given on Figure 1:

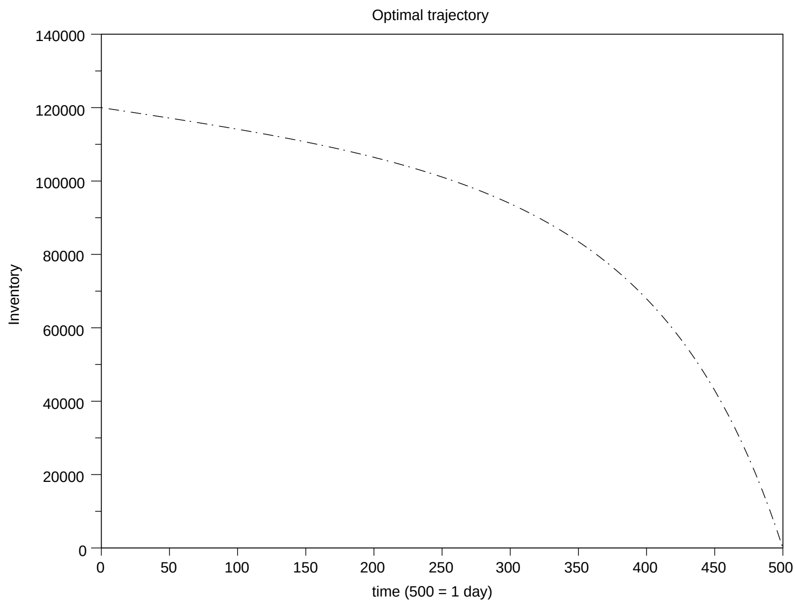

Figure 1: Example of a trading curve for a Target Close algorithm. $q_0 = 120000$, $\sigma = 0.6$, $\eta = 0.08$, $\phi = 0.5$, $V_t = V = 1200000$ and $\gamma = 6 \times 10^{-6}$.

1.3 Premium for guaranteed close

The above theorem solves the problem of the optimal trajectory but it is only one part of the problem. We indeed defined above the reserve price $\ell_{TC}(q_0)$ charged by an intermediary who would propose a service in which the price S_T is guaranteed. We know that:

$$\ell_{TC}(q_0) = -\Phi(q_0) + \inf_{q \in AC_{q_0,0}(0,T)} \mathcal{I}^{TC}(q).$$

Using Proposition 2, this equation becomes:

$$\ell_{TC}(q_0) = -\Phi(q_0) + \inf_{q \in AC_{q_0,0}(0,T)} \mathcal{I}^{\tilde{IS}}(q).$$

Using the optimal trajectory q^*, we get:

$$\ell_{TC}(q_0) = -\Phi(q_0) + \mathcal{I}^{TC}(q^*).$$

From a numerical point of view, since the solution of the Hamiltonian system characterizing the minimizer q^* of $\mathcal{I}^{TC}$ can be approximated easily using a Newton scheme, there is no problem to approximate $\ell_{TC}(q_0)$. However, it would be interesting to have a closed-form expression for $\ell_{TC}(q_0)$. In fact, using Proposition 2, we see that there is a closed-form expression for $\ell_{TC}(q_0)$ if and only if there is a closed-form expression for a block trade in the case of an IS order. There are in fact two cases in which we can have a closed-form expression. The first case is when $\phi = 1$ and V_t is constant. The second case is when we approximate $\mathcal{I}^{TC}(q^*)$ by its value when $T \to +\infty$, still in the flat volume case $V_t = V$.

Proposition 3. *Let us consider the special case $\phi = 1$ where $V_t = V$ is constant. In that case:*

$$\ell_{TC}(q_0) = -\Phi(q_0) - \psi|q_0| + \sqrt{\frac{\eta\gamma\sigma^2}{2V}} \frac{1}{2} \frac{\sinh\left(2\sqrt{\frac{\gamma\sigma^2 V}{2\eta}}T\right)}{\sinh^2\left(\sqrt{\frac{\gamma\sigma^2 V}{2\eta}}T\right)} q_0^2$$

Proof:

We know for [18] and Proposition 2 that the optimal trajectory q^* is given by:

$$q^*(t) = q_0 \left(1 - \frac{\sinh\left(\sqrt{\frac{\gamma\sigma^2 V}{2\eta}}t\right)}{\sinh\left(\sqrt{\frac{\gamma\sigma^2 V}{2\eta}}T\right)}\right)$$

Now,

$$\ell_{TC}(q_0) = -\Phi(q_0) - \psi|q_0| + \int_0^T \left(\eta \frac{q^{*\prime}(t)^2}{V} + \frac{1}{2}\gamma\sigma^2(q_0 - q^*(t))^2\right) ds$$

$$\begin{aligned}
\ell_{TC}(q_0) &= -\Phi(q_0) - \psi|q_0| + \int_0^T \left(\eta \frac{q^{*\prime}(t)^2}{V} + \frac{1}{2}\gamma\sigma^2(q_0 - q^*(t))^2\right) ds \\
&= -\Phi(q_0) - \psi|q_0| + \frac{1}{2}\gamma\sigma^2 \frac{q_0^2}{\sinh^2\left(\sqrt{\frac{\gamma\sigma^2 V}{2\eta}}T\right)} \int_0^T \left(\cosh^2\left(\sqrt{\frac{\gamma\sigma^2 V}{2\eta}}t\right)\right. \\
&\quad \left. + \sinh^2\left(\sqrt{\frac{\gamma\sigma^2 V}{2\eta}}t\right)\right) dt
\end{aligned}$$

$$= \quad -\Phi(q_0) - \psi|q_0| + \frac{1}{2}\gamma\sigma^2 \frac{q_0^2}{\sinh^2\left(\sqrt{\frac{\gamma\sigma^2 V}{2\eta}}T\right)} \int_0^T \cosh\left(2\sqrt{\frac{\gamma\sigma^2 V}{2\eta}}t\right) dt$$

$$= \quad -\Phi(q_0)dz - \psi|q_0| + \sqrt{\frac{\eta\gamma\sigma^2}{2V}} \frac{1}{2} \frac{\sinh\left(2\sqrt{\frac{\gamma\sigma^2 V}{2\eta}}T\right)}{\sinh^2\left(\sqrt{\frac{\gamma\sigma^2 V}{2\eta}}T\right)} q_0^2.$$

$\square$

Now, the second case is just an approximation. It states (see [18] and Proposition 2) that when $V_t = V$ is a constant,

$$\inf_{q \in AC_{q_0,0}(0,T)} \mathcal{I}^{TC}(q) = \inf_{q \in AC_{q_0,0}(0,T)} \mathcal{I}^{\tilde{IS}}(q) \simeq \int_0^{q_0} H^{-1}\left(\frac{\gamma\sigma^2}{2V}x^2\right) dx$$

where H is the Legendre transform of L defined in Theorem 1 and where H^{-1} is the inverse of $H : \mathbb{R}_+ \to \mathbb{R}_+$.

This approximation corresponds to the limit case $T \to \infty$ and we have seen in [18] with realistic values that it is a rather good approximation for T equal to 1 day. In all cases, it is a lower bound to the actual premium.

Appendix to Section 1. Optimal strategies can be considered deterministic.

We now prove for the sake of completeness that no strategy in $\mathcal{A}$ can do better than a strategy in $\mathcal{A}_{det}$.

Proposition 4.

$$\sup_{v \in \mathcal{A}} \mathbb{E}\left[-\exp\left(-\gamma(X_T - q_0 S_T)\right)\right] = \sup_{v \in \mathcal{A}_{\text{det}}} \mathbb{E}\left[-\exp\left(-\gamma(X_T - q_0 S_T)\right)\right].$$

Proof:

For any $v \in \mathcal{A}$, we know that

$$\mathbb{E}\left[-\exp\left(-\gamma(X_T - q_0 S_T)\right)\right] = -\exp\left(-\gamma\Phi(q_0)\right)$$

$$\times\mathbb{E}\left[\exp\left(\gamma\left(V_s L\left(\frac{v_s}{V_s}\right) + \psi|v_s|\right)\right)\exp\left(-\gamma\sigma\int_0^T (q_s - q_0)dW_s\right)\right].$$

Hence, if we introduce the probability measure $\mathbb{Q}$ defined by the Radon-Nikodym derivative (we can apply Girsanov theorem since q is bounded):

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\gamma\sigma\int_0^T (q_s - q_0)dW_s - \frac{1}{2}\gamma^2\sigma^2\int_0^T (q_s - q_0)^2 ds\right),$$

then:

$$\mathbb{E}\left[-\exp\left(-\gamma(X_T - q_0 S_T)\right)\right] = -\exp\left(-\gamma \Phi(q_0)\right)$$
$$\times \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\gamma \int_0^T \left(V_s L\left(\frac{v_s}{V_s}\right) + \psi|v_s|\right) ds\right) \exp\left(\frac{1}{2}\gamma^2\sigma^2 \int_0^T (q_s - q_0)^2 ds\right)\right].$$

Hence:

$$\mathbb{E}\left[-\exp\left(-\gamma(X_T - q_0 S_T)\right)\right] = \mathbb{E}^{\mathbb{Q}}\left[-\exp\left(-\gamma\left(\Phi(q_0) - \mathcal{I}^{TC}(q)\right)\right)\right].$$

Now, let us fix $\omega \in \Omega$. We have that $t \mapsto q_t(\omega) \in AC_{q_0,0}([0,T])$ almost surely and then $\mathcal{I}^{TC}(q(\omega)) \geq \mathcal{I}^{TC}(q^*)$ almost surely. This leads to

$$\mathbb{E}\left[-\exp\left(-\gamma(X_T - q_0 S_T)\right)\right] \leq -\exp\left(-\gamma\left(\Phi(q_0) - \mathcal{I}^{TC}(q^*)\right)\right),$$

i.e.:

$$\mathbb{E}\left[-\exp\left(-\gamma(X_T - q_0 S_T)\right)\right] \leq \sup_{v \in \mathcal{A}_{\text{det}}} \mathbb{E}\left[-\exp\left(-\gamma(X_T - q_0 S_T)\right)\right].$$

We then obtain

$$\sup_{v \in \mathcal{A}} \mathbb{E}\left[-\exp\left(-\gamma(X_T - q_0 S_T)\right)\right] \leq \sup_{v \in \mathcal{A}_{\text{det}}} \mathbb{E}\left[-\exp\left(-\gamma(X_T - q_0 S_T)\right)\right].$$

Since the converse inequality holds, the result is proved. □

2 The model with closing auction

To start with, we recall that on Euronext, there is a closing auction at the end of the continuous auction. The volume transacted at the closing auction represents an important part of the overall volume. To put figures on this assertion, we plotted the volume at the closing auction for three stocks (Gaz de France, BNP Paribas, and Pernod-Ricard). In addition to these volumes (left scale), we plotted the percentage they represented in the overall daily volume (right scale):

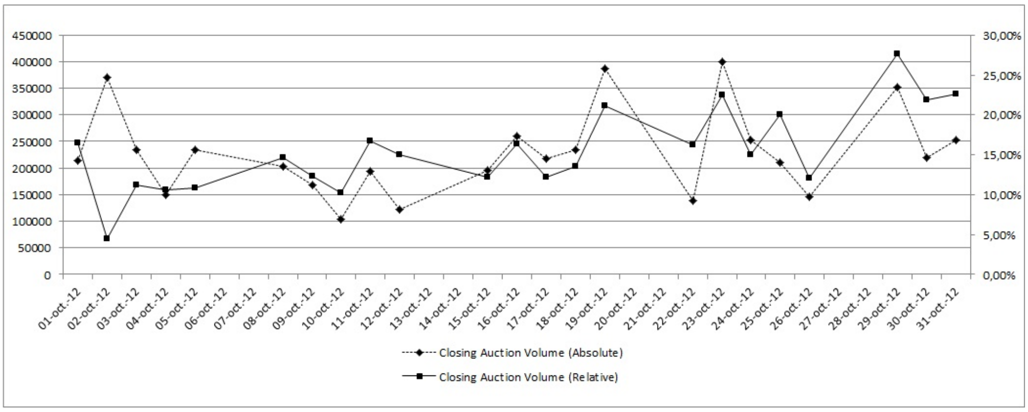

Figure 2: Volume at the closing auction for Gaz de France

We see on the above figures that the volume available during the closing auction is large and highly random. Moreover, predicting the volume during the closing

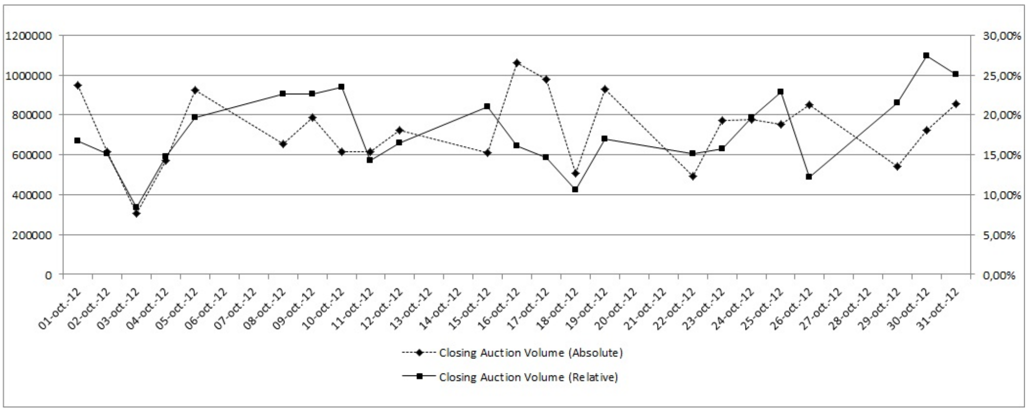

Figure 3: Volume at the closing auction for BNP Paribas

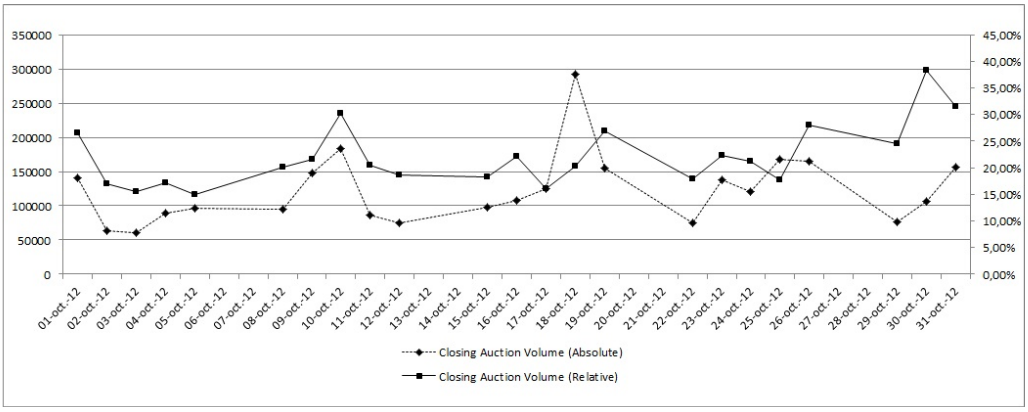

Figure 4: Volume at the closing auction for Pernod Ricard

auction is a difficult task. We see indeed that the volume at the close measured as a proportion of the total volume of the day is also highly random. Hence, the volume during the continuous auction hardly explains the volume at the close. The consequence is that, if one keeps a high volume to be executed during the closing auction, then, there is a risk to impact the closing price. One may think that this is not a problem since one wants to be executed at the closing price. However, one wants to be executed as close as possible to the closing price if and only if the closing price remains an exogenous benchmark that is only slightly modified by its own trades.

This being said, the basic idea of some practitioners is to keep a small quantity of shares to be executed during the closing auction, this small quantity being determined as a quantile of the distribution of the volume at the close. The problem of a target close strategy is indeed that the quantity to be executed during the closing auction must be decided in advance and not during the closing auction: when the continuous auction ends, the remaining quantity to be executed must be executed during the auction! This idea of a fixed volume to be executed during the auction is the main ingredient of our first (simple) model.
To model the closing auction, a simple setup is the following. At the end of the continuous auction, that is at time T, the price is S_T. The remaining quantity in the portfolio is denoted q_{close} and this quantity is executed at a price $S_{T'}$ defined as:

$$S_{T'} = S_T - g(q_{close}) + \sigma_{close}\tilde{\epsilon},$$

where:

- $q_{close} \mapsto g(q_{close})$ is an increasing function, positive for positive q_{close}, and negative for negative q_{close}.
- σ_{close} is the standard deviation of the difference between the price at the end of the continuous auction and the closing price.
- $\tilde{\epsilon}$ is a $\mathcal{N}(0,1)$ random variable, independent of the Brownian motion W.

2.1 A simple model

The first model we consider is a model in which q_{close} is determined exogenously, for instance by a quantile of the distribution of the volume at the closing auction. In this simple model (we consider $|q_0| \geq |q_{close}|$, otherwise the problem is trivial), the objective function is:

$$\mathbb{E}\left[-\exp(-\gamma(X_{T'} - q_0 S_{T'}))\right],$$

and the cash process is given by:

$$X_{T'} = \int_0^T \left(v_s S_s - V_s L\left(\frac{v_s}{V_s}\right) - \psi|v_s|\right) ds + q_{close} S_{T'}.$$

Obviously, we now optimize on $v \in \mathcal{A}_{q_0 - q_{close}, det}$, where:

$$\mathcal{A}_{q_0 - q_{close}, det} = \left\{ v \in L^\infty(0,T), \int_0^T v_s ds = q_0 - q_{close} \right\}.$$

We start with the distribution of $X_{T'} - q_0 S_{T'}$:

Proposition 5. *Let us consider $v \in \mathcal{A}_{q_0 - q_{close}, det}$.*
We have:

$$X_{T'} - q_0 S_{T'} = \Phi(q_0 - q_{close}) - \int_0^T V_s L\left(\frac{v_s}{V_s}\right) ds - \psi \int_0^T |v_s| ds$$

$$+ \int_0^T \sigma(q_s - q_0) dW_s + (q_0 - q_{close}) g(q_{close}) - (q_0 - q_{close}) \sigma_{close} \tilde{\epsilon}.$$

Hence,

$$X_{T'} - q_0 S_{T'} \sim \mathcal{N}\left(\Phi(q_0 - q_{close}) - \int_0^T V_s L\left(\frac{v_s}{V_s}\right) ds - \psi \int_0^T |v_s| ds \right.$$

$$\left. + (q_0 - q_{close}) g(q_{close}), \sigma^2 \int_0^T (q_s - q_0)^2 ds + \sigma_{close}^2 (q_0 - q_{close})^2 \right).$$

Proof:

By definition:

$$\begin{aligned}
X_{T'} - q_0 S_{T'} &= (q_{close} - q_0) S_{T'} + \int_0^T v_s S_s ds - \int_0^T V_s L\left(\frac{v_s}{V_s}\right) ds - \psi \int_0^T |v_s| ds \\
&= (q_{close} - q_0)(S_{T'} - S_T) + \Phi(q_0 - q_{close}) + \int_0^T \sigma(q_s - q_0) dW_s \\
&\quad - \int_0^T V_s L\left(\frac{v_s}{V_s}\right) ds - \psi \int_0^T |v_s| ds \\
&= \Phi(q_0 - q_{close}) - \int_0^T V_s L\left(\frac{v_s}{V_s}\right) ds - \psi \int_0^T |v_s| ds + \int_0^T \sigma(q_s - q_0) dW_s \\
&\quad + (q_0 - q_{close}) g(q_{close}) - (q_0 - q_{close}) \sigma_{close} \tilde{\epsilon}.
\end{aligned}$$

Now, since q is deterministic, we get the announced distribution for $X_{T'} - q_0 S_{T'}$. $\square$

Using the same reasoning as in Section 1, we obtain that the problem boils down to minimizing the following function:

$$\begin{aligned}
\mathcal{I}_{close}^{TC} \; : \; AC_{q_0, q_{close}}(0,T) &\to \mathbb{R}_+ \\
q &\mapsto \int_0^T \left(V_s L\left(\tfrac{q'(s)}{V_s}\right) + \psi |q'(s)| + \tfrac{1}{2} \gamma \sigma^2 (q_0 - q(s))^2 \right) ds,
\end{aligned}$$

Now, using the same change of variables as for Proposition 2, we have:

Proposition 6. *Let us define $\widehat{V} : t \in [0,T] \mapsto V_{T-t}$. Then q^* is a minimizer of $\mathcal{I}^{TC}_{close}$ if and only if $\tilde{q}^* : t \in [0,T] \mapsto q_0 - q^*(T-t)$ is a minimizer of:*

$$\begin{array}{rcccl} \mathcal{I}^{\tilde{IS}}_{close} & : & AC_{q_0-q_{close},0}(0,T) & \to & \mathbb{R}_+ \\ & & \tilde{q} & \mapsto & \int_0^T \left(\widehat{V}_s L\left(\frac{\tilde{q}'(s)}{\widehat{V}_s}\right) + \psi|\tilde{q}'(s)| + \frac{1}{2}\gamma\sigma^2\tilde{q}(s)^2\right) ds, \end{array}$$

This leads straightforwardly to the following theorem:

Theorem 2. *Let us denote $\tilde{q}^*$ the unique minimizer of $\mathcal{I}^{\tilde{IS}}_{close}$, that is the IS curve to liquidate $q_0 - q_{close}$ shares.*
The optimal trajectory in our first model with auction is:

$$q^*(t) = q_0 - \tilde{q}^*(T-t) - q_{close}\delta_{T'}.$$

The premium for guaranteed close is:

$$\begin{aligned} \ell_{TC}(q_0) &= -\Phi(q_0 - q_{close}) - (q_0 - q_{close})g(q_{close}) + \frac{1}{2}\sigma^2_{close}(q_0 - q_{close})^2 \\ &\quad + \inf_{q \in AC_{q_0-q_{close},0}(0,T)} \mathcal{I}^{\tilde{IS}}_{close}(q). \end{aligned}$$

2.2 Making q_{close} a function of q_0

The above model is simple since the quantity to be executed at the close is decided in advance, independently of q_0. The problem with this first model is that liquidating $q_0 - q_{close}$ may lead to execution costs that could be avoided if q_{close} was larger, without impacting that much the price during the closing auction. Hence, when q_0 is large, it may be interesting to choose q_{close} as a function of q_0. However, in such a case, the best benchmark price may not be $S_{T'}$ anymore. If indeed we choose $S_{T'}$ as a benchmark, then we do not really care about the impact at the closing auction. The model we propose consists in considering a convex combination of S_T and $S_{T'}$ as a benchmark.

In other words, we have to solve the following optimization problem:

$$\sup_{q_{close}, v \in \mathcal{A}_{q_0-q_{close},det}} \mathbb{E}\left[-\exp(-\gamma(X_{T'} - q_0(\pi S_{T'} + (1-\pi)S_T)))\right],$$

where $\pi \in [0,1]$.

We start with the distribution of $X_{T'} - q_0(\pi S_{T'} + (1-\pi)S_T)$.

Proposition 7. *Let us consider $v \in \mathcal{A}_{q_0-q_{close},det}$.*
We have:

$$X_{T'} - q_0(\pi S_{T'} + (1-\pi)S_T) = \Phi(q_0 - q_{close}) - \int_0^T V_s L\left(\frac{v_s}{V_s}\right) ds - \psi \int_0^T |v_s| ds$$

$$+\int_0^T \sigma(q_s - q_0)dW_s + (\pi q_0 - q_{close})g(q_{close}) - (\pi q_0 - q_{close})\sigma_{close}\tilde{\epsilon}.$$

Hence,

$$X_{T'} - q_0(\pi S_{T'} + (1-\pi)S_T) \sim \mathcal{N}\left(\Phi(q_0 - q_{close}) - \int_0^T V_s L\left(\frac{v_s}{V_s}\right)ds - \psi\int_0^T |v_s|ds\right.$$

$$\left. +(\pi q_0 - q_{close})g(q_{close}), \sigma^2\int_0^T (q_s - q_0)^2 ds + \sigma_{close}^2(\pi q_0 - q_{close})^2\right).$$

Proof:

We have $X_{T'} - q_0(\pi S_{T'} + (1-\pi)S_T) = X_{T'} - q_0 S_{T'} + (1-\pi)q_0(S_{T'} - S_T)$. Hence, the result follows from Proposition 5. □

To avoid dynamic arbitrage between the continuous auction and the closing auction and to simplify exposition, we consider the specific case where $f = k$ is a constant (or equivalently $\Phi(q) = \frac{k}{2}q^2$) and $g(q) = kq$. We also consider that $\psi = 0$. In that case, the problem boils down to minimizing the following function:

$$\begin{array}{rcccl} \mathcal{I}_{close,\pi}^{TC} & : & AC_{q_0}(0,T) & \rightarrow & \mathbb{R}_+ \\ & & q & \mapsto & \int_0^T \left(V_s L\left(\frac{q'(s)}{V_s}\right) + \frac{\gamma}{2}\sigma^2(q_0 - q(s))^2\right)ds \\ & & & & +kq_0 q(T)(1-\pi) + \frac{k}{2}q(T)^2 + \frac{\gamma}{2}\sigma_{close}^2(\pi q_0 - q(T))^2, \end{array}$$

where $AC_{q_0}(0,T)$ is the set of absolutely continuous function q on $[0,T]$ with $q(0) = q_0$.

Since the dependence on $q(T)$ is convex, the Hamiltonian characterization for this problem is given by the following Proposition:

Proposition 8. *There exists a unique minimizer $q^* \in AC_{q_0}(0,T)$ of the function $\mathcal{I}_{close,\pi}^{TC}$, characterized by:*

$$\left\{\begin{array}{rcl} p'(t) & = & \gamma\sigma^2(q^*(t) - q_0) \\ q^{*\prime}(t) & = & V_t H'(p(t)) \end{array}\right. ,$$

where $q^*(0) = q_0$, $\quad p(T) = -kq_0(1-\pi) - kq(T) - \gamma\sigma_{close}^2(q(T) - \pi q_0)$.

To better understand the model and because in that case we have a closed form solution, we consider the special case where $\phi = 1$ and where $V_t = V$ is constant.

Proposition 9. *Let us consider the special case where $\phi = 1$ and where $V_t = V$ is constant. We have:*

$$q^*(t) = q_0\left(1 - A\sinh\left(\sqrt{\frac{\gamma\sigma^2 V}{2\eta}}t\right)\right),$$

where

$$A = \frac{\frac{kV}{2\eta}(2-\pi) + \frac{\gamma\sigma_{close}^2 V}{2\eta}(1-\pi)}{\sqrt{\frac{\gamma\sigma^2 V}{2\eta}}\cosh\left(\sqrt{\frac{\gamma\sigma^2 V}{2\eta}}T\right) + \frac{kV}{2\eta}\sinh\left(\sqrt{\frac{\gamma\sigma^2 V}{2\eta}}T\right) + \frac{\gamma\sigma_{close}^2 V}{2\eta}\sinh\left(\sqrt{\frac{\gamma\sigma^2 V}{2\eta}}T\right)}$$

Proof:

In the case we consider, the above system reduces to the ordinary differential equation:

$$q^{*\prime\prime}(t) = \frac{\gamma\sigma^2 V}{2\eta} q^*(t),$$

with $q^*(0) = q_0 \quad q^{*\prime}(T) = -\frac{kV}{2\eta}(q(T) + (1-\pi)q_0) - \frac{\gamma\sigma_{close}^2 V}{2\eta}(q(T) - \pi q_0)$.
The unique solution of this equation is of the form $q^*(t) = q_0\left(1 - A\sinh\left(\sqrt{\frac{\gamma\sigma^2 V}{2\eta}}t\right)\right)$, where A is given in the Proposition. □

A few words have to be said with respect to this result. First, the quantity $q(T)$ must be liquidated at the close. This quantity $q(T)$ is proportional to q_0 (this is more realistic than a fixed constant) and it is increasing with π. It means that the quantity executed at the closing auction is larger when the benchmark is closer to the post-auction price. This is in line with the intuition.
An interesting point is that, although one may not feel comfortable with choosing π, the figures obtained for $\pi = 0$ and $\pi = 1$ give lower and upper bounds for the quantity to be liquidated at the closing price.

The premium for guaranteed closing price can be computed as in the preceding subsection once $q_{close} = q^*(T)$ has been computed.

Conclusion

Some investors are willing to see their orders executed as close as possible to the closing price. For that purpose, they can use a Target Close strategy. In this paper, we discuss the way to design Target Close strategies and we show, in the absence of closing auction, that there is a correspondence between IS strategies and Target Close strategies. When there is a closing auction, we developed two new models. The first one gives exogenously the quantity to be executed during the closing auction. The second one however makes the quantity executed at the close a function of the size of the initial order.

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