IT'S ALL RELATIVE: MEAN FIELD GAME EXTENSIONS OF MERTON'S PROBLEM

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Introduction

Merton's portfolio problem

Merton (1969)

- Merton (following Samuelson) built a reference model for optimal consumption and investment choices.
- Used Hamilton-Jacobi-Bellman (HJB) equation.
- Various settings with closed-form solutions:
  - CRRA utility function.
  - CARA utility function (raises the question of negative consumption).
  - Finite and infinite horizon.

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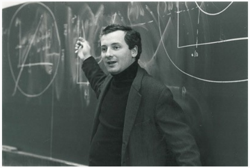
One of Merton's great successes with the Black-Scholes-Merton formula.



A vast literature

Many extensions

- Transaction costs
- Taxes
- Labor income
- Stochastic volatility
- Trading constraints
- Habit formation preferences
- Recursive utility
- Partial information
- ...

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Several mathematical methods

- HJB equation
- Dual / Martingale method

Our contribution

Goal

- Introduction of comparison / jealousy / competition in Merton's portfolio problem.
- Showing Jean-Michel Lasry an example of MFG Master equation that could be solved in closed form.
  Remark: MFG of controls $\rightarrow$ *not only characterized by a Master equation.*

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- Showing Jean-Michel Lasry an example of MFG Master equation that could be solved in closed form.

  Remark: MFG of controls $\rightarrow$ not only characterized by a Master equation.

Literature I knew on competition in optimal investment

- Static model: Guéant, Lasry, Lions, Mean Field Games and Applications, Paris-Princeton Lectures on Mathematical Finance 2010.
- Dynamic model: Espinosa, Touzi, Optimal Investment under Relative Performance Concerns, Mathematical Finance, 2015.

Our contribution

Literature I discovered in August (Discussion with René Carmona on MFG)

- Lacker, Zariphopoulou, Mean field and n-agent games for optimal investment under relative performance criteria. Mathematical Finance, 2017.
- Lacker, Soret, Many-player games of optimal consumption and investment under relative performance criteria. arXiv 28 May 2019.

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Our framemork

- Less general:
  - A single common stock (the motivation was a Master equation).
  - Agents differ only by their wealth (original MFG framework).
  - No N-player game, only MFG.
- More general:
  - CARA case (raises the question of negative consumption).
  - Recursive utility case (Epstein-Zin/Duffie-Epstein/Duffie-Lions).

The model

Merton's portfolio problem

Original problem

- Stock: $dS_t = \mu S_t dt + \sigma S_t dW_t$.
- Wealth: $dX_t = (r + \theta_t(\mu - r))X_t dt + \theta_t \sigma X_t dW_t - c_t dt$.
- Optimization problem:

$$\sup_{c,\theta} \mathbb{E}\left[\int_0^T e^{-\rho t} u(c_t) dt + \epsilon e^{-\rho T} u(X_T)\right].$$

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HJB equation

$$\begin{aligned} 0 \quad = \quad & \partial_t V - \rho V + rx\partial_x V + \sup_c \{u(c) - c\partial_x V\} \\ & + \sup_\theta \left\{\theta(\mu - r)x\partial_x V + \frac{1}{2}\theta^2\sigma^2 x^2 \partial_{xx}^2 V\right\} \end{aligned}$$

Terminal condition: $V(T, x) = \epsilon u(x)$.

Solution

CARA case $u(x) = -\frac{1}{\gamma} e^{-\gamma x}$

- Ansatz: $V(t,x) = -\frac{1}{\gamma} e^{-\gamma f(t)x + g(t)}$.
- Resulting equations:

$$\begin{aligned} 0 &= f' + rf - f^2, \quad f(T) = 1. \\ 0 &= g' - fg + f - f\log(f) - \rho - \frac{(\mu - r)^2}{2\sigma^2}, \quad g(T) = \log(\epsilon). \end{aligned}$$

- Optimal controls (feedback form):

$$\begin{aligned} c^*(t,x) &= f(t)x - \frac{1}{\gamma}\left(\log(f(t)) + g(t)\right). \\ M^*(t,x) &= \theta^*(t,x)x = \frac{\mu - r}{\gamma\sigma^2 f(t)}. \end{aligned}$$

Solution

CRRA case $u(x) = \frac{1}{1-\gamma} x^{1-\gamma}$

- Ansatz: $V(t,x) = \frac{1}{1-\gamma} f(t) x^{1-\gamma}$.
- Resulting equation (Bernoulli):

$$f' = \left(\rho - (1-\gamma)\left(r + \frac{(\mu - r)^2}{2\gamma\sigma^2}\right)\right) f - \gamma f^{-\frac{1-\gamma}{\gamma}}$$

  with terminal condition $f(T) = \epsilon$.
- Optimal controls (feedback form):

$$\begin{aligned} c^*(t,x) &= f(t)^{-\frac{1}{\gamma}} x. \\ \theta^*(t,x) &= \frac{\mu - r}{\gamma\sigma^2}. \end{aligned}$$

Introducing competition I – the CARA case

The CARA case

- A population of agents with the same preferences (wealth is distributed).
- A common stock: $dS_t = \mu S_t dt + \sigma S_t dW_t$.
- Wealth: $dX_t = rX_t dt + (\mu - r) M_t dt + \sigma M_t dW_t - c_t dt$.
- Optimization problem:

$$\sup_{c,\theta} \mathbb{E}\left[\int_0^T -e^{-\rho t}\frac{1}{\gamma}e^{-\gamma(c_t - \beta \overline{c}_t)}dt - \epsilon e^{-\rho T}\frac{1}{\gamma}e^{-\gamma(X_T - \beta \overline{X}_T)}\right],$$

where $\beta \in [0, 1)$ and $(\overline{c}, \overline{X})$ designates averages in the population.

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where $\beta \in [0,1)$ and $(\overline{c}, \overline{X})$ designates averages in the population.

MFG approach

- Given dynamics of averages, solve HJB to obtain optimal controls.
- Find dynamics of averages consistent with optimal controls (fixed-point problem).

Towards a solution

Dynamics of the averages

$$\begin{aligned} \overline{c}_t &= \overline{c}(t, \overline{X}_t). \\ d\overline{X}_t &= \overline{\mu}(t, \overline{X}_t)dt + \overline{\sigma}(t, \overline{X}_t)dW_t - \overline{c}(t, \overline{X}_t)dt. \end{aligned}$$

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HJB equation

$$\begin{aligned} 0 &= \partial_t V - \rho V + rx\partial_x V + \sup_c \left\{ -\frac{1}{\gamma} e^{-\gamma(c - \beta \overline{c}(t, \overline{x}))} - c\partial_x V \right\} \\ &+ \sup_M \left\{ (\mu - r)M\partial_x V + \frac{1}{2}\sigma^2 M^2 \partial^2_{xx} V + \sigma\overline{\sigma}(t, \overline{x}) M \partial^2_{x\overline{x}} V \right\} \\ &+ (\overline{\mu}(t, \overline{x}) - \overline{c}(t, \overline{x}))\partial_{\overline{x}} V + \frac{1}{2}\overline{\sigma}(t, \overline{x})^2 \partial^2_{\overline{x}\overline{x}} V \end{aligned}$$

Terminal condition: $V(T, x, \overline{x}) = -\frac{\epsilon}{\gamma} e^{-\gamma(x - \beta\overline{x})}$.

Optimal controls and equilibrium conditions

- Ansatz: $V(t,x,\overline{x}) = -\frac{1}{\gamma} e^{-\gamma f(t)(x-\beta\overline{x})+g(t)}$.
- Optimal controls given averages (feedback form):

$$
\begin{aligned}
c^*(t,x,\overline{x}) &= f(t)(x-\beta\overline{x}) + \beta\overline{c}(t,\overline{x}) - \frac{1}{\gamma}\left(\log(f(t)) + g(t)\right). \\
M^*(t,x,\overline{x}) &= \frac{\mu - r}{\gamma\sigma^2 f(t)} + \beta\frac{\overline{\sigma}(t,\overline{x})}{\sigma}.
\end{aligned}
$$

- Equilibrium equations:

$$
\begin{aligned}
\overline{c}(t,\overline{x}) &= f(t)\overline{x} - \frac{1}{(1-\beta)\gamma}\left(\log(f(t)) + g(t)\right). \\
\overline{\sigma}(t,\overline{x}) &= \frac{\mu - r}{(1-\beta)\gamma\sigma f(t)}. \\
\overline{\mu}(t,\overline{x}) &= r\overline{x} + \frac{(\mu - r)^2}{(1-\beta)\gamma\sigma^2 f(t)}.
\end{aligned}
$$

Solution

- Resulting equations (exactly the same):

$$\begin{aligned} 0 &= f' + rf - f^2, \quad f(T) = 1. \\ 0 &= g' - fg + f - f\log(f) - \rho - \frac{(\mu - r)^2}{2\sigma^2}, \quad g(T) = \log(\epsilon). \end{aligned}$$

- Optimal controls (feedback form):

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\end{aligned}
$$

Everything works as if the risk aversion γ had been replaced by $(1-\beta)\gamma$: agents take more risk with competition in the CARA case!

Introducing competition II – the CRRA case

The CRRA case

- A population of agents with the same preferences (wealth is distributed).
- A common stock: $dS_t = \mu S_t dt + \sigma S_t dW_t$.
- Wealth: $dX_t = (r + \theta_t(\mu - r))X_t dt + \sigma\theta_t X_t dW_t - c_t dt$.
- Optimization problem:

$$\sup_{c,\theta} \mathbb{E}\left[\int_0^T e^{-\rho t}\frac{1}{1-\gamma}\left(\frac{c_t}{\overline{c}_t^{\beta}}\right)^{1-\gamma} dt + \epsilon e^{-\rho T}\frac{1}{1-\gamma}\left(\frac{X_T}{\overline{X}_T^{\beta}}\right)^{1-\gamma}\right],$$

where $\beta \in [0,1]$ and $(\overline{c}, \overline{X})$ designates averages in the population.

Remark: the averages are arithmetic averages, not geometric as in Lacker-Zariphopoulou and Lacker-Soret. This is made possible by the fact that agents have the same preferences and invest in the same asset.

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As above we use a MFG approach.

Toward a solution

Dynamics of the averages

$$\begin{aligned} \overline{c}_t &= \alpha(t)\overline{X}_t. \\ d\overline{X}_t &= \overline{\mu}\overline{X}_t dt + \overline{\sigma}\overline{X}_t dW_t - \alpha(t)\overline{X}_t dt. \end{aligned}$$

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HJB equation

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0 \quad = \quad & \partial_t V - \rho V + rx\partial_x V + \sup_c \left\{ \frac{1}{1-\gamma}\left(\frac{c}{\alpha^\beta \overline{x}^\beta}\right)^{1-\gamma} - c\partial_x V \right\} \\
& + \sup_\theta \left\{ \theta(\mu - r)x\partial_x V + \frac{1}{2}\theta^2\sigma^2 x^2 \partial^2_{xx} V + \theta\sigma\overline{\sigma}x\overline{x}\partial^2_{x\overline{x}} V \right\} \\
& + (\overline{\mu} - \alpha(t))\overline{x}\partial_{\overline{x}} V + \frac{1}{2}\overline{\sigma}^2\overline{x}^2\partial^2_{\overline{x}\overline{x}} V
\end{aligned}
$$

Terminal condition: $V(T, x, \overline{x}) = \frac{\epsilon}{1-\gamma}\left(\frac{x}{\overline{x}^\beta}\right)^{1-\gamma}$.

Optimal controls and equilibrium conditions

- Ansatz: $V(t,x,\overline{x}) = \frac{1}{1-\gamma} f(t) \left(\frac{x}{\overline{x}^{\beta}}\right)^{1-\gamma}$.
- Optimal controls given averages (feedback form):

$$
\begin{aligned}
c^*(t,x,\overline{x}) &= \alpha(t)^{\beta\left(1-\frac{1}{\gamma}\right)} f(t)^{-\frac{1}{\gamma}} x. \\
\theta^*(t,x,\overline{x}) &= \frac{\mu - r}{\gamma\sigma^2} - \beta\frac{1-\gamma}{\gamma}\frac{\overline{\sigma}(t,\overline{x})}{\sigma}.
\end{aligned}
$$

- Equilibrium equations:

$$
\begin{aligned}
\alpha(t) &= f(t)^{-\frac{1}{\gamma+\beta(1-\gamma)}}. \\
\overline{\sigma} &= \frac{\mu - r}{(\gamma + \beta(1-\gamma))\sigma}. \\
\overline{\mu} &= r + \frac{(\mu - r)^2}{(\gamma + \beta(1-\gamma))\sigma^2}.
\end{aligned}
$$

Solution

- Resulting equation (Bernoulli):

$$\begin{aligned} f'(t) &= \left(\rho - (1-\gamma)(1-\beta)\left(r + \frac{(\mu - r)^2}{2(\gamma + \beta(1-\gamma))\sigma^2}\right)\right) f(t) \\ &\quad - (\gamma + \beta(1-\gamma))\, f(t)^{-\frac{(1-\beta)(1-\gamma)}{\gamma+\beta(1-\gamma)}}. \end{aligned}$$

  with terminal condition $f(T) = \epsilon$.

- Optimal controls (feedback form):

$$\begin{aligned} c^*(t,x) &= f(t)^{-\frac{1}{\gamma+\beta(1-\gamma)}} x. \\ \theta^*(t,x) &= \frac{\mu - r}{(\gamma + \beta(1-\gamma))\sigma^2}. \end{aligned}$$

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Everything works as if the risk aversion γ had been replaced by $\gamma + \beta(1-\gamma) = \beta + (1-\beta)\gamma$: competition makes agents behave more like if they had a log utility function.

Recursive utility

Epstein-Zin / Kreps-Porteus

- Recursive utility to disentangle risk aversion γ and intertemporal elasticity of substitution (IES) ψ (CRRA case: $\psi = \frac{1}{\gamma}$).
- Discrete-time version in the no-competition case:

$$U_t = \left[\rho c_t^{1-\frac{1}{\psi}} + (1-\rho)\mathbb{E}\left[U_{t+1}^{1-\gamma} \right]^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right]^{\frac{1}{1-\frac{1}{\psi}}}$$

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Equivalent aggregator

The continuous-time counterpart has been studied by Duffie-Epstein / Duffie-Lions who obtained an equivalent "aggregator":

$$(c, V) \mapsto \frac{\rho}{1-\frac{1}{\psi}} \frac{c^{1-\frac{1}{\psi}} - ((1-\gamma)V)^{\frac{1-\frac{1}{\psi}}{1-\gamma}}}{((1-\gamma)V)^{\frac{1-\frac{1}{\psi}}{1-\gamma}-1}}.$$

Introducing competition III – the recursive utility case

Recursive utility formulation

- A population of agents with the same preferences (wealth is distributed).
- A common stock: $dS_t = \mu S_t dt + \sigma S_t dW_t$.
- Wealth: $dX_t = (r + \theta_t(\mu - r))X_t dt + \sigma\theta_t X_t dW_t - c_t dt$.
- Optimization problem:

$$V_t = \sup_{c,\theta} \mathbb{E}\left[\int_t^T \frac{\rho}{1-\frac{1}{\psi}} \frac{\left(\frac{c_s}{\overline{c}_s^{\beta}}\right)^{1-\frac{1}{\psi}} - ((1-\gamma)V_s)^{\frac{1-\frac{1}{\psi}}{1-\gamma}}}{((1-\gamma)V_s)^{\frac{1-\frac{1}{\psi}}{1-\gamma}-1}} ds \right.$$

$$\left. + \frac{(\rho\epsilon)^{\frac{1-\gamma}{1-\frac{1}{\psi}}}}{1-\gamma} \left(\frac{X_T}{\overline{X}_T^{\beta}}\right)^{1-\gamma} \right],$$

where $\beta \in [0,1]$ and $(\overline{c}, \overline{X})$ designates averages in the population.

Toward a solution

Dynamics of the averages

$$\begin{aligned} \overline{c}_t &= \alpha(t)\overline{X}_t. \\ d\overline{X}_t &= \overline{\mu}\overline{X}_t dt + \overline{\sigma}\overline{X}_t dW_t - \alpha(t)\overline{X}_t dt. \end{aligned}$$

Toward a solution

Dynamics of the averages

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HJB equation

$$\begin{aligned}0 \;&=\; \partial_t V + rx\partial_x V + \sup_c \left\{ \frac{\rho}{1-\frac{1}{\psi}} \frac{\left(\frac{c}{\alpha(t)^{\beta}\overline{x}^{\beta}}\right)^{1-\frac{1}{\psi}}}{((1-\gamma)V)^{\frac{1-\frac{1}{\psi}}{1-\gamma}-1}} - c\partial_x V \right\}\\ &-\; \rho\frac{1-\gamma}{1-\frac{1}{\psi}}V + \sup_\theta \left\{ \theta(\mu - r)x\partial_x V + \frac{1}{2}\theta^2\sigma^2 x^2 \partial^2_{xx} V + \theta\sigma\overline{\sigma}x\overline{x}\partial^2_{x\overline{x}} V \right\}\\ &+\; (\overline{\mu} - \alpha(t))\overline{x}\partial_{\overline{x}} V + \frac{1}{2}\overline{\sigma}^2\overline{x}^2\partial^2_{\overline{x}\overline{x}} V\end{aligned}$$

Terminal condition: $V(T, x, \overline{x}) = \frac{(\rho\epsilon)^{\frac{1-\gamma}{1-\frac{1}{\psi}}}}{1-\gamma}\left(\frac{x}{\overline{x}^{\beta}}\right)^{1-\gamma}.$

Optimal controls and equilibrium conditions

- Ansatz: $V(t,x,\overline{x}) = \frac{1}{1-\gamma} f(t) \left(\frac{x}{\overline{x}^{\beta}}\right)^{1-\gamma}$.
- Optimal controls given averages (feedback form):

$$
\begin{aligned}
c^*(t,x,\overline{x}) &= \rho^{\psi}\alpha(t)^{\beta(1-\psi)} f(t)^{\frac{1-\psi}{1-\gamma}} x.\\
\theta^*(t,x,\overline{x}) &= \frac{\mu - r}{\gamma\sigma^2} - \beta\frac{1-\gamma}{\gamma}\frac{\overline{\sigma}(t,\overline{x})}{\sigma}.
\end{aligned}
$$

- Equilibrium equations:

$$
\begin{aligned}
\alpha(t) &= \rho^{\frac{1}{\frac{1}{\psi}+\beta\left(1-\frac{1}{\psi}\right)}} f(t)^{-\frac{1-\frac{1}{\psi}}{1-\gamma}\frac{1}{\frac{1}{\psi}+\beta(1-\frac{1}{\psi})}}.\\
\overline{\sigma} &= \frac{\mu - r}{(\gamma + \beta(1-\gamma))\sigma}.\\
\overline{\mu} &= r + \frac{(\mu - r)^2}{(\gamma + \beta(1-\gamma))\sigma^2}.
\end{aligned}
$$

Solution (valid for instance if $\gamma < 1 < \psi$ or $\psi < 1 < \gamma$)

- Resulting equation (Bernoulli):

$$f'(t) = \left(\frac{1-\gamma}{1-\frac{1}{\psi}}\rho - (1-\gamma)(1-\beta)\left(r + \frac{(\mu-r)^2}{2(\gamma+\beta(1-\gamma))\sigma^2}\right)\right) f(t)$$

$$- \frac{1-\gamma}{1-\frac{1}{\psi}}\left(\frac{1}{\psi} + \beta\left(1-\frac{1}{\psi}\right)\right)\rho^{\frac{1}{\frac{1}{\psi}+\beta\left(1-\frac{1}{\psi}\right)}} f(t)^{1-\frac{1-\frac{1}{\psi}}{1-\gamma}\frac{1}{\frac{1}{\psi}+\beta\left(1-\frac{1}{\psi}\right)}}.$$

with terminal condition $f(T) = (\rho\epsilon)^{\frac{1-\gamma}{1-\frac{1}{\psi}}}$.

- Optimal controls (feedback form):

$$\begin{aligned} c^*(t,x) &= \rho^{\frac{1}{\frac{1}{\psi}+\beta\left(1-\frac{1}{\psi}\right)}} f(t)^{-\frac{1-\frac{1}{\psi}}{1-\gamma}\frac{1}{\frac{1}{\psi}+\beta(1-\frac{1}{\psi})}} x. \\ \theta^*(t,x) &= \frac{\mu-r}{(\gamma+\beta(1-\gamma))\sigma^2}. \end{aligned}$$

Solution (valid for instance if $\gamma < 1 < \psi$ or $\psi < 1 < \gamma$)

Different writing

If $\gamma_\beta = \gamma + \beta(1-\gamma)$ and $\frac{1}{\psi_\beta} = \frac{1}{\psi} + \beta\left(1 - \frac{1}{\psi}\right)$ then

$$f'(t) = \left(\frac{1-\gamma_\beta}{1-\frac{1}{\psi_\beta}}\rho - (1-\gamma_\beta)\left(r + \frac{(\mu - r)^2}{2\gamma_\beta\sigma^2}\right)\right) f(t)$$

$$- \frac{1-\gamma_\beta}{1-\frac{1}{\psi_\beta}}\frac{1}{\psi_\beta}\rho^{\psi_\beta} f(t)^{1-\frac{1-\frac{1}{\psi_\beta}}{1-\gamma_\beta}\psi_\beta}, \qquad f(T) = (\rho\epsilon)^{\frac{1-\gamma_\beta}{1-\frac{1}{\psi_\beta}}}.$$

Optimal controls (feedback form):

$$c^*(t,x) = \rho^{\psi_\beta} f(t)^{-\frac{1-\frac{1}{\psi_\beta}}{1-\gamma_\beta}\psi_\beta} x, \qquad \theta^*(t,x) = \frac{\mu - r}{\gamma_\beta\sigma^2}.$$

Solution (valid for instance if $\gamma < 1 < \psi$ or $\psi < 1 < \gamma$)

Different writing

If $\gamma_\beta = \gamma + \beta(1-\gamma)$ and $\frac{1}{\psi_\beta} = \frac{1}{\psi} + \beta\left(1 - \frac{1}{\psi}\right)$ then

$$f'(t) = \left(\frac{1-\gamma_\beta}{1-\frac{1}{\psi_\beta}}\rho - (1-\gamma_\beta)\left(r + \frac{(\mu-r)^2}{2\gamma_\beta\sigma^2}\right)\right) f(t) - \frac{1-\gamma_\beta}{1-\frac{1}{\psi_\beta}}\frac{1}{\psi_\beta}\rho^{\psi_\beta} f(t)^{1-\frac{1-\frac{1}{\psi_\beta}}{1-\gamma_\beta}\psi_\beta}, \qquad f(T) = (\rho\epsilon)^{\frac{1-\gamma_\beta}{1-\frac{1}{\psi_\beta}}}.$$

Optimal controls (feedback form):

$$c^*(t,x) = \rho^{\psi_\beta} f(t)^{-\frac{1-\frac{1}{\psi_\beta}}{1-\gamma_\beta}\psi_\beta} x, \qquad \theta^*(t,x) = \frac{\mu - r}{\gamma_\beta\sigma^2}.$$

Everything works as if the risk aversion γ and the IES ψ had been replaced by γ_β and ψ_β. Competition makes agents behave more like if they had a log utility function in terms of risk and IES equal to 1.

Conclusion

What we found

- Closed-form solutions for Merton's problem with competition in the CARA, CRRA, and recursive utility cases.
- Examples of MFG of controls with common noise that could be solved in closed form.

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What's next

Generalizing Lacker-Soret (with different agent types) in the case of Epstein-Zin recursive utility:

Olivier Guéant, Competition in Merton's problem: the recursive utility case.

Questions

Thanks for your attention.
Questions.