A Uniqueness Result for Mean Field Games

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Abstract

In this text, we provide a condition for the mean field games equations to have a unique solution. This result proved by J.-M. Lasry and P.L. Lions in 2010 and presented in [4] generalizes the uniqueness result proved in [2, 3] and allows to consider Hamiltonian functions that appear in certain congestion models.

Introduction

In this text we consider the mean field games equations introduced in [1, 2, 3] for a general Hamiltonian. These partial differential equations, hereafter denoted (MFG) are considered on the domain $[0,T]\times\Omega$, Ω standing for $(0,1)^d$ and consist in the following equations:

$$\text{(HJB)} \qquad \partial_t u - \rho u + \frac{\sigma^2}{2}\Delta u + \mathcal{H}(x,\nabla u, m) = 0$$

$$\text{(K)} \qquad \partial_t m + \nabla\cdot(m\partial_p\mathcal{H}(x,\nabla u,m)) = \frac{\sigma^2}{2}\Delta m$$

with:

- u and m 1-periodic
- Terminal condition: $u(T,\cdot) = u_T(\cdot)$ a given terminal payoff
- Initial condition: $m(0,\cdot) = m_0(\cdot)$ a given initial probability distribution function

In their seminal papers ([2, 3]), J.M. Lasry and P.L. Lions presented a uniqueness result for the above equations that we restate here:

Theorem 1 (Uniqueness result for a separable Hamiltonian)**.** *Assume that $\mathcal{H}(x,p,m) = H(x,p) + f(x,m)$ with:*

- $\forall x\in\Omega, p\in\mathbb{R}^d \mapsto H(x,p)$ *a convex function*
- $\forall M>0, x\in\Omega\mapsto \sup_{|p|\leq M}|H(x,p)|\in L^1(\Omega)$
- $f:\Omega\times\mathcal{P}(\Omega)\mapsto\mathbb{R}$ *verifying:*

$$\int_\Omega (f(x,m_1)-f(x,m_2))\,d(m_1-m_2)(x)\geq 0 \Rightarrow m_1 = m_2$$

Then, there exists at most one solution (u,m) of (MFG) with[1] *u and m in $C^{1,2}([0,T]\times\bar{\Omega})$.*

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[1]In particular, m will be understood as a probability distribution function.

In this text, we will consider Hamiltonian functions of the form $\mathcal{H}(x,p,m) = H(x,p,m(x))$ where m is a probability distribution function. This framework allows to consider Hamiltonian functions that appear in congestion problems such as $\frac{|\nabla u|^a}{m^b}$.

We are going to prove the following theorem:

Theorem 2 (Uniqueness)**.** *Assume that for absolutely continuous measures $\mathcal{H}(x,p,m) = H(x,p,m(x))$ with:*

- $\forall x \in \Omega, (p,z) \mapsto H(x,p,z)$ *a C^2 function*
- $\forall x \in \Omega, \forall p \in \mathbb{R}^d, \forall z \in \mathbb{R}^+$*:*

$$\begin{pmatrix} z\partial^2_{pp}H(x,p,z) & \frac{1}{2}z\partial^2_{pz}H(x,p,z) \\ \frac{1}{2}z\left(\partial^2_{pz}H(x,p,z)\right)' & -\partial_z H(x,p,z) \end{pmatrix} > 0$$

Then, there exists at most one solution (u,m) of (MFG) with u and m in $C^{1,2}([0,T]\times\bar{\Omega})$.

Proof:
Let's consider two smooth solutions (u^1,m^1) and (u^0,m^0).
We consider the same strategy as in [2]:

$$\begin{aligned}
& \frac{d}{dt}\int_\Omega (u^1-u^0)(m^1-m^0) \\
= & \int_\Omega (\partial_t u^1 - \partial_t u^0)(m^1-m^0) + \int_\Omega (u^1-u^0)(\partial_t m^1 - \partial_t m^0) \\
= & \int_\Omega \rho(u^1-u^0)(m^1-m^0) - \frac{\sigma^2}{2}\int_\Omega (\Delta u^1 - \Delta u^0)(m^1-m^0) \\
- & \int_\Omega (H(x,\nabla u^1,m^1) - H(x,\nabla u^0,m^0))(m^1-m^0) + \frac{\sigma^2}{2}\int_\Omega (u^1-u^0)(\Delta m^1 - \Delta m^0) \\
+ & \int_\Omega (\nabla u^1 - \nabla u^0)\cdot\left(m^1\partial_p H(x,\nabla u^1,m^1) - m^0\partial_p H(x,\nabla u^0,m^0)\right) \\
= & \int_\Omega \rho(u^1-u^0)(m^1-m^0) - \int_\Omega (H(x,\nabla u^1,m^1) - H(x,\nabla u^0,m^0))(m^1-m^0) \\
+ & \int_\Omega (\nabla u^1 - \nabla u^0)\cdot\left(m^1\partial_p H(x,\nabla u^1,m^1) - m^0\partial_p H(x,\nabla u^0,m^0)\right)
\end{aligned}$$

Now, let's call $u^\theta = u^0 + \theta(u^1-u^0)$ and $m^\theta = m^0 + \theta(m^1-m^0)$ and let's define

$$\mathcal{I}(\theta) = -\int_\Omega (H(x,\nabla u^\theta,m^\theta) - H(x,\nabla u^0,m^0))(m^\theta - m^0)$$

$$+\int_\Omega (\nabla u^\theta - \nabla u^0)\cdot\Big(m^\theta\partial_p H(x,\nabla u^\theta,m^\theta) - m^0\partial_p H(x,\nabla u^0,m^0)\Big)$$

Clearly,

$$\mathcal{I}(\theta) = \theta\Bigg[-\int_\Omega (H(x,\nabla u^\theta,m^\theta) - H(x,\nabla u^0,m^0))(m^1 - m^0)$$

$$+\int_\Omega (\nabla u^1 - \nabla u^0)\cdot\Big(m^\theta\partial_p H(x,\nabla u^\theta,m^\theta) - m^0\partial_p H(x,\nabla u^0,m^0)\Big)\Bigg]$$

Hence $\left.\frac{\mathcal{I}(\theta)}{\theta}\right|_{\theta=0} = 0$.

Now,

$$
\begin{aligned}
\frac{d}{d\theta}\left(\frac{\mathcal{I}(\theta)}{\theta}\right) &= -\int_\Omega (m^1-m^0)\partial_p H(x,\nabla u^\theta, m^\theta)\cdot(\nabla u^1-\nabla u^0) \\
&- \int_\Omega (m^1-m^0)^2 \partial_z H(x,\nabla u^\theta, m^\theta) \\
&+ \int_\Omega (m^1-m^0)\partial_p H(x,\nabla u^\theta, m^\theta)\cdot(\nabla u^1-\nabla u^0) \\
&+ \int_\Omega (\nabla u^1-\nabla u^0)' \left(m^\theta \partial^2_{pp} H(x,\nabla u^\theta, m^\theta)\right)(\nabla u^1-\nabla u^0) \\
&+ \int_\Omega (m^1-m^0) m^\theta \partial^2_{pz} H(x,\nabla u^\theta, m^\theta)\cdot(\nabla u^1-\nabla u^0)
\end{aligned}
$$

$$
= \int_\Omega \begin{pmatrix} \nabla u^1-\nabla u^0 \\ m^1-m^0 \end{pmatrix}' \begin{pmatrix} m^\theta \partial^2_{pp} H(x,\nabla u^\theta, m^\theta) & \frac{1}{2} m^\theta \partial^2_{pz} H(x,\nabla u^\theta, m^\theta) \\ \frac{1}{2} m^\theta \left(\partial^2_{pz} H(x,\nabla u^\theta, m^\theta)\right)' & -\partial_z H(x,\nabla u^\theta, m^\theta) \end{pmatrix} \begin{pmatrix} \nabla u^1-\nabla u^0 \\ m^1-m^0 \end{pmatrix}
$$

From the positivity of the matrix we get that $\mathcal{I}(1) \geq 0$ and hence:

$$
\frac{d}{dt}\int_\Omega (u^1-u^0)(m^1-m^0) - \rho \int_\Omega (u^1-u^0)(m^1-m^0) \geq 0
$$

$$
\Rightarrow \frac{d}{dt}\left[e^{-\rho t}\int_\Omega (u^1-u^0)(m^1-m^0)\right] \geq 0
$$

But $m^0(0,\cdot) = m^1(0,\cdot)$ and $u^0(T,\cdot) = u^1(T,\cdot)$. Hence, $\int_\Omega (u^1-u^0)(m^1-m^0)$ must be 0 for all times and this is possible only if $\begin{pmatrix} \nabla u^1-\nabla u^0 \\ m^1-m^0 \end{pmatrix}$ is equal to nought for all times. Hence, $m^0 = m^1$ and straightforwardly $u^0 = u^1$. $\square$

References

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