Tournament-Induced Risk-Shifting: 
A Mean Field Games Approach.

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Abstract

The agency problem between an investor and his mutual funds managers has long been studied in the economic literature. Because the very business of mutual funds managers is not only to manage money but also, and rather, to increase the money under management, one of the numerous agency problems is the implicit incentive induced by the relationship between inflows and performance. If the consequences of incentives, be they implicit or explicit – as for compensation schemes of individual asset managers –, are well known in terms of risk-shifting when the incentives are linked to a benchmark, the very fact that the mutual fund market is a tournament does not seem to be modeled properly in the literature. In this paper, we propose a mean field games model to quantify the risk-shifting induced by a tournament-like competition between mutual funds.

Introduction

Investors would like their money to be invested so as to maximize the risk-adjusted expected return of their portfolio. However, asset managers can have other incentives in addition to satisfying their clients. The resulting agency problems are numerous and, for most of them, well studied in the economic literature. Individual asset managers may indeed have explicit incentives that induce risk-shifting. These explicit incentives are well instanced by the compensation schemes usually observed in the industry and a huge literature is dedicated to this question, either to optimally design the incentive schemes (see for example Cadenillas, Cvitanic and Zapatero [5] or Dybvig, Farnsworth and Carpenter [13]) or to study the impact of incentive contracts (such as those, option-based, observed in practice) on the managers’ decisions (see for example Carpenter [6], Hodder and Jackwerth [17], Hugonnier and Kaniel [19] or Ross [28]). Several empirical studies, such as Brown, Harlow and Starks [4], Chen and Pennacchi [7] or Sirri and Tufano [30], insist on the relevance of these individual incentives to explain behaviors that may differ across managers due to past performance. A common observation is that, after the first six months of each year, the funds that best performed tend to decrease the volatility of their strategies for the second half of the year. Also, these empirical studies insist on the importance of implicit incentives at the individual scale as it is exemplified by the impact of career concerns (see Arora and Ou-Yang [1] and Chevallier and Ellison [10]), individual wage targets or reputation concerns (see Huddart [18]) on trading behavior.

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But this literature, linked to the corporate finance literature on stock options for executives, only gives part of the picture since it mainly considers the impact in terms of risk, or in terms of equilibrium prices (see Basak and Pavlova [2], Cuoco and Kaniel [11] or Vayanos and Woolley [31]), of incentives directly targeted at the individual fund manager level. In practice, several incentives must be considered at the larger scale of the mutual fund. Asset managers’ compensations indeed depend on the performance fees received by the mutual fund and asset managers therefore have an incentive linked to the high-water marks when such schemes are present (see Goetzmann, Ingersoll and Ross [15] for the risk-shifting due to performance fee contracts with high-water mark provisions). More generally, an implicit incentive arises from the so-called flow-performance relationship (see Basak, Pavlova and Shapiro [3] and Chevalier and Elison [8]).

Because mutual funds compete for customers, they must understand the reason for inflows and outflows in the fund and so do asset managers whose remunerations are linked to the financial health of their firm. The mechanisms behind this flow-performance relationship are empirically discussed and the two main hypotheses are a tournament effect in which investors favor the best performing mutual funds and a benchmark effect in which investors favor mutual funds that beat a commonly used benchmark. The latter effect is well modeled but the former still need proper modeling (although there exists a literature on tournaments in contract theory for instance and applied to labor market – see Lazear and Rosen [24] for such an application), certainly because of the difficulty to deal mathematically with ranking effects.

The purpose of this paper, is to propose a model to quantify the risk-shifting due to this tournament-induced incentive and we are going to deal with a theoretical model that marginally modifies the 2-period Markowitz framework and embeds the willingness of asset managers to attract new customers. This willingness is modeled by a tournament effect in the sense that asset managers maximize a convex combination of a risk-adjusted expected return and of their ranking among asset managers.

In the first section, we are going to present the model in a general framework. The second section will be dedicated to a resolution that generalizes the well-known case of the CARA utility function Markowitz framework. The third section will discuss the actual risk-shifting on numerical examples.

The model

We consider a continuum of asset managers. All asset managers have the same amount under management at time 0 and are going to choose the portfolio in which they invest. The possible portfolios are made of any convex combinations of the two available assets. We consider indeed a market with two assets: there is a risk-free asset whose return is denoted \( r \) (the weight on this asset in the portfolio will be \( 1 - \theta \)) and a risky asset whose return is \( r + \tilde{\epsilon} \) where \( \tilde{\epsilon} \) is a random variable that is central in what follows (the weight on this asset in the portfolio will be \( \theta \)).

The criterion used to choose the portfolio has two components. First, there is a pure Markowitz part where the asset manager is going to consider an expected utility maximization. Second, since there is an incentive for the asset manager not only to please the current clients but also to maximize the number of clients or, quite similarly, the total

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1The only article that tests these two hypotheses one against the other is, to our knowledge, the article by Reed and Wu [27] but the hypothesis tested is based on the median and not on the true ranking or cumulative distribution function.

2see Markowitz [25] or Merton [26] for the usual framework, either in discrete or in continuous time.

3In fact, although mutual fund investors seem to use raw return performance, Del Guercio and Tkac [12] show that investors flock disproportionately to recent winners but do not withdraw assets from recent losers, a fact that is confirmed by Gruber [16] (favoring a more complex model than ours)
amount under management, there is a competition between asset managers. Therefore, each asset manager wants to signal that he is better than his peers and he is going to maximize, in addition to the usual utility criterion, his ranking inside the asset managers community (this is the tournament-induced incentive discussed in the introduction).

As a consequence, each asset manager is going to maximize, according to his beliefs, the following expression:

$$E[u(X) + \beta \tilde{R}]$$

where $X = 1 + r + \theta \tilde{\epsilon}$ is the wealth at date 1, where $\beta$ is a constant that models the relative importance of the tournament component, and where $\tilde{R}$ is the random variable that ranks asset managers. This variable $\tilde{R}$ takes values in $[0, 1]$, 0 being associated to the worst-performing asset manager at the end of the period and 1 being for the best-performing asset manager. In other words, this is simply the cumulative distribution function (even though it is a random variable) of the returns in the population of asset managers.

So far, asset managers were similar to one another. However, even if they have at time 0 the same amount 1 under management, we are going to suppose that they have different beliefs about the random variable $\tilde{\epsilon}$ that models the excess return of the risky asset. We say that a given asset manager is of the $\epsilon$-type if he believes that $\tilde{\epsilon} \sim N(\epsilon, \sigma^2)$ where the variance $\sigma^2$ is supposed to be the same for all asset managers. In other words, we suppose that they agree on the volatility of the risky asset but they have different beliefs on its expected return. In what follows $f$ will stand for the distribution of the types $\epsilon$ and it will be supposed to be symmetric around 0 ($f$ is even) for exposition purpose.

**Resolution**

To solve the problem and compare the result to the well known Markowitz case without competition, we are going to consider the case of a CARA utility function, that is $u(x) = -\exp(-\lambda x)$.

Let’s consider an $\epsilon$-type asset manager. We are going to derive the first order condition that characterizes his optimal $\theta$.

**Proposition 1** ($FOC_\epsilon$). The first order condition associated to an $\epsilon$-type asset manager is:

$$\text{(FOC}_\epsilon \text{)} - \lambda^2 \sigma^2 \left( \theta - \frac{\epsilon}{\lambda \sigma^2} \right) \exp \left( -\lambda(1 + r) - \lambda \theta \epsilon + \frac{1}{2} \lambda^2 \theta^2 \sigma^2 \right) + \beta m(\theta) C(\epsilon) = 0$$

where $m$ stands for the probability distribution function of the $\theta$’s at equilibrium, where $C(\cdot) = 2 \left[ N \left( \frac{\cdot}{\sigma} \right) - \frac{1}{2} \right]$ is an odd function, positive on $\mathbb{R}_+$, $N$ being the cumulative distribution function of a gaussian variable $N(0, 1)$.

**Proof:**

An $\epsilon$-type asset manager maximizes:

$$E_\epsilon \left[ u(1 + r + \theta \tilde{\epsilon}) + \beta \tilde{R} \right]$$

It’s easy to see that $\tilde{R} = 1_{\tilde{\epsilon} > 0} M(\theta) + 1_{\tilde{\epsilon} \leq 0} (1 - M(\theta))$ where $M$ stands for the cumulative distribution function of the weights $\theta$. 
Also,
\[
E_\epsilon [u(1 + r + \theta \tilde{\epsilon})] = -E_\epsilon [\exp (-\lambda (1 + r + \theta \tilde{\epsilon}))]
\]
\[
= -\exp \left( -\lambda (1 + r + \theta \epsilon) + \frac{1}{2} \lambda^2 \theta^2 \sigma^2 \right)
\]

Hence, the optimal \( \theta \) is given by the argmax of:
\[
-\exp \left( -\lambda (1 + r + \theta \epsilon) + \frac{1}{2} \lambda^2 \theta^2 \sigma^2 \right) + \beta E_\epsilon [1_{\tilde{\epsilon} > 0} M(\theta) + 1_{\tilde{\epsilon} \leq 0} (1 - M(\theta))]
\]

Let’s differentiate the above equation. We get the first order condition for an \( \epsilon \)-type asset manager:
\[
-\lambda^2 \sigma^2 \left( \theta - \frac{\epsilon}{\lambda \sigma^2} \right) \exp \left( -\lambda (1 + r) - \lambda \theta \epsilon + \frac{1}{2} \lambda^2 \theta^2 \sigma^2 \right) + \beta E_\epsilon [1_{\tilde{\epsilon} > 0} - 1_{\tilde{\epsilon} \leq 0}] m(\theta) = 0
\]

But,
\[
\mathbb{P}_\epsilon (\tilde{\epsilon} > 0) - \mathbb{P}_\epsilon (\tilde{\epsilon} \leq 0)
\]
\[
= 2 \left[ \mathbb{P}_\epsilon (\tilde{\epsilon} > 0) - \frac{1}{2} \right]
\]
\[
= 2 \left[ \mathbb{P} \left( \mathcal{N}(0, 1) > -\frac{\epsilon}{\sigma} \right) - \frac{1}{2} \right]
\]
\[
= C(\epsilon)
\]

Hence we get the result. □

**Proposition 2** (Differential equation for \( \epsilon \mapsto \theta(\epsilon) \)). Let’s consider the function \( \epsilon \mapsto \theta(\epsilon) \) that gives the optimal \( \theta \) for each type. If \( \theta \) is is a continuously differentiable function then it verifies the following differential equation:
\[
-\lambda^2 \sigma^2 \left( \theta - \frac{\epsilon}{\lambda \sigma^2} \right) \exp \left( -\lambda (1 + r) - \lambda \theta \epsilon + \frac{1}{2} \lambda^2 \theta^2 \sigma^2 \right) \frac{d\theta}{d\epsilon} + \beta f(\epsilon) C(\epsilon) = 0 \quad (\star)
\]

Moreover, \( \theta \) must verify \( \theta(0) = 0 \).

**Proof:**

To go from the distribution \( f \) of the types to the distribution \( m \) of the \( \theta \)'s, we need a coherence equation that simply is:
\[
m(\theta) \theta'(\epsilon) = f(\epsilon)
\]

Now, if we take \( (FOC_\epsilon) \) and multiply by \( \theta'(\epsilon) \) we get the ODE \( (\star) \).

Now, because \( C(0) = 0 \), the equation \( (FOC_0) \) is simply
\[
-\lambda^2 \sigma^2 \theta \exp \left( -\lambda (1 + r) + \frac{1}{2} \lambda^2 \theta^2 \sigma^2 \right) = 0
\]

and the unique solution of this equation is \( \theta = 0 \). □

To simplify the analysis, let’s highlight that the differential equation satisfies the following property that allows us to work only with \( \epsilon > 0 \):
Proposition 3. Let $\theta$ be a solution of the ODE $(\ast)$. Then, $\overline{\theta}$ defined by $\overline{\theta}(\epsilon) = -\theta(-\epsilon)$ is also a solution of $(\ast)$.

Proof:

Let’s write the ODE for $-\epsilon$:

$$-\lambda^2 \sigma^2 \left( \theta(-\epsilon) + \frac{\epsilon}{\lambda \sigma^2} \right) \exp \left( -\lambda (1 + r) + \lambda \theta(-\epsilon) \epsilon + \frac{1}{2} \lambda^2 \theta(-\epsilon)^2 \sigma^2 \right) \theta'(-\epsilon) + \beta f(-\epsilon) C(-\epsilon) = 0$$

Since $f$ is even and $C$ is odd, we get:

$$-\lambda^2 \sigma^2 \left( -\overline{\theta}(\epsilon) + \frac{\epsilon}{\lambda \sigma^2} \right) \exp \left( -\lambda (1 + r) - \lambda \overline{\theta}(\epsilon) \epsilon + \frac{1}{2} \lambda^2 \overline{\theta}(\epsilon)^2 \sigma^2 \right) \overline{\theta}'(\epsilon) - \beta f(\epsilon) C(\epsilon) = 0$$

Now, let’s go back to the risk analysis. First of all, we want $\theta$ to be an increasing function of $\epsilon$ (this is simply because being optimistic about the return of the risky asset should lead to buy more of this asset than being pessimistic about it). Consequently, for $\epsilon > 0$, the ODE implies that

$$\theta(\epsilon) > \frac{\epsilon}{\lambda \sigma^2}$$

Therefore, if we note $\epsilon \mapsto \theta_0(\epsilon) = \epsilon \frac{\lambda}{\lambda \sigma^2}$ the usual solution of the Markowitz problem without ranking, then we should have $\forall \epsilon > 0, \theta(\epsilon) > \theta_0(\epsilon) > 0$ and symmetrically $\forall \epsilon < 0, \theta(\epsilon) < \theta_0(\epsilon) < 0$. This means that the ranking effect induces a higher risk exposure.

Now, we are ready to write the exact problem we need to solve. We are looking for a function $\epsilon \mapsto \theta(\epsilon)$ defined for $\epsilon > 0$ (the odd function associated to this branch will be our solution) that verifies:

$$\theta'(\epsilon) = \frac{\beta C(\epsilon) f(\epsilon)}{\lambda^2 \sigma^2 \exp(-\lambda (1 + \theta(\epsilon) \epsilon + \frac{1}{2} \lambda^2 \theta(\epsilon)^2 \sigma^2))} \frac{1}{\theta(\epsilon) - \theta_0(\epsilon)}$$

and the two additional conditions:

- $\theta(\epsilon) > \theta_0(\epsilon) = \frac{\epsilon}{\lambda \sigma^2}$
- $\lim_{\epsilon \to 0} \theta(\epsilon) = 0$

Proposition 4 (Existence and Uniqueness). There exists a unique function that verifies the equation $(\ast)$ with the two additional constraints:

- $\theta(\epsilon) > \theta_0(\epsilon) = \frac{\epsilon}{\lambda \sigma^2}$
- $\lim_{\epsilon \to 0} \theta(\epsilon) = 0$
Proof:

If there exists one, let’s consider a solution \( \theta \) of the problem and let’s introduce the function \( z \) defined by:

\[
z(\epsilon) = (\theta(\epsilon) - \theta_0(\epsilon))^2
\]

If we want to inverse this equation and get \( \theta \) as a function of \( z \) then we get:

\[
\theta(\epsilon) = \theta_0(\epsilon) \pm \sqrt{z(\epsilon)}
\]

but since \( \theta(\epsilon) > \theta_0(\epsilon) \) we clearly can inverse the equation and get:

\[
\theta(\epsilon) = \theta_0(\epsilon) + \sqrt{z(\epsilon)} := \Theta(\epsilon, z(\epsilon))
\]

Now, if we differentiate the equation that defines \( z \) we have:

\[
z'(\epsilon) = 2 \left( \theta'(\epsilon) - \theta'_0(\epsilon) \right) (\theta(\epsilon) - \theta_0(\epsilon))
\]

\[
= \frac{2\beta C(\epsilon)f(\epsilon)}{\lambda^2\sigma^2 \exp(-\lambda(1 + r + \Theta(\epsilon, z(\epsilon))\epsilon) + \frac{1}{2}\lambda^2\sigma^2\Theta(\epsilon, z(\epsilon))^2)} - \frac{2}{\lambda\sigma^2} (\theta(\epsilon) - \theta_0(\epsilon))
\]

Now, if we consider the following ODE:

\[
z'(\epsilon) = \frac{2\beta C(\epsilon)f(\epsilon)}{\lambda^2\sigma^2 \exp(-\lambda(1 + r + \Theta(\epsilon, z(\epsilon))\epsilon) + \frac{1}{2}\lambda^2\sigma^2\Theta(\epsilon, z(\epsilon))^2)} - \frac{2}{\lambda\sigma^2} \sqrt{z(\epsilon)}
\]

with \( z(0) = 0 \), there exists a local solution to this equation by Cauchy-Peano and this solution defined on a neighborhood \( V \) of 0 verifies:

\[
z'(\epsilon) + \frac{2}{\lambda\sigma^2} \sqrt{z(\epsilon)} > 0, \quad \forall \epsilon \in V \cap \mathbb{R}^*_+, \quad z(0) = 0
\]

From this we can deduce that \( z \) is strictly positive on \( V \cap \mathbb{R}^*_+ \). Consider indeed a minimum of \( z \) on \( V \cap \mathbb{R}^*_+ \). If the minimum were reached in \( \epsilon^0 \neq 0 \) then \( z'(\epsilon^0) \leq 0 \) and hence \( z(\epsilon^0)_+ > 0 \). But this is not possible since \( z(0) = 0 \). Thus \( z \) is positive on \( V \cap \mathbb{R}^*_+ \). Now if \( z \) were equal to 0 for a given \( \epsilon > 0 \), then \( z'(\epsilon) > 0 \) and hence \( z \) would be strictly negative for some \( \epsilon \in (0, \epsilon) \) and this has been proved to be false.

Hence, there exists a (positive) local solution\(^4\) \( z \) of

\[
z'(\epsilon) = \frac{2\beta C(\epsilon)f(\epsilon)}{\lambda^2\sigma^2 \exp(-\lambda(1 + r + \Theta(\epsilon, z(\epsilon))\epsilon) + \frac{1}{2}\lambda^2\sigma^2\Theta(\epsilon, z(\epsilon))^2)} - \frac{2}{\lambda\sigma^2} \sqrt{z(\epsilon)}
\]

with \( z(0) = 0 \).

Now, this solution must be unique because

\[
(\epsilon, z) \in (\mathbb{R}^*_+)^2 \mapsto \frac{2\beta C(\epsilon)f(\epsilon)}{\lambda^2\sigma^2 \exp(-\lambda(1 + r + \Theta(\epsilon, z(\epsilon))\epsilon) + \frac{1}{2}\lambda^2\sigma^2\Theta(\epsilon, z(\epsilon))^2)} - \frac{2}{\lambda\sigma^2} \sqrt{z}
\]

defines a decreasing function with respect to \( z \).

Now, we can locally (for \( \epsilon \geq 0 \)) define \( \theta \) as \( \theta(\epsilon) = \theta_0(\epsilon) + \sqrt{z(\epsilon)} \). Since there is no problem outside of \( \epsilon = 0 \) (i.e. the Cauchy-Lipschitz theorem can be applied directly to any

\(^4\)For \( \epsilon \geq 0 \)
point in the domain \( \{(\epsilon, \theta)| \epsilon > 0, \theta > \frac{\epsilon}{\sqrt{\sigma^2}}\} \) the uniqueness is proved and we need to prove that the solution is defined on the whole domain.

Imagine our solution is not defined on the whole domain and let us note \( \tau \) the upper bound of the definition interval. Since \( \theta \) is increasing, we have either:

\[
\lim_{\epsilon \to \tau} \theta(\epsilon) = +\infty
\]

or

\[
\lim_{\epsilon \to \tau} \theta(\epsilon) = \theta_0(\tau)
\]

We are going to show that these two cases are impossible. Imagine first that \( \lim_{\epsilon \to \tau} \theta(\epsilon) = +\infty \). Then, we can suppose there exists an interval \((\epsilon, \tau)\) such that \( \forall \epsilon \in (\epsilon, \tau), \theta(\epsilon) > \theta_0(\epsilon) + 1 \). Hence, on \((\epsilon, \tau)\) we have:

\[
\theta'(\epsilon) \leq \frac{\beta C(\epsilon)f(\epsilon)}{\lambda^2 \sigma^2} \exp\left(-\lambda(1 + r + \theta(\epsilon)) + \frac{1}{2} \lambda^2 \sigma^2 \theta(\epsilon)^2\right)
\]

But \( \lambda \theta(\epsilon) - \frac{1}{2} \lambda^2 \sigma^2 \theta(\epsilon)^2 \leq \frac{\epsilon^2}{2 \sigma^2} \) so that:

\[
\forall \epsilon \in (\epsilon, \tau), \theta'(\epsilon) \leq \frac{\beta C(\epsilon)f(\epsilon)}{\lambda^2 \sigma^2} \exp\left(\lambda(1 + r) + \frac{\epsilon^2}{2 \sigma^2}\right)
\]

Hence,

\[
\forall \epsilon \in (\epsilon, \tau), \theta(\epsilon) \leq \theta(\epsilon) + \int_{\epsilon}^\tau \frac{\beta C(\xi)f(\xi)}{\lambda^2 \sigma^2} \exp\left(\lambda(1 + r) + \frac{\xi^2}{2 \sigma^2}\right) d\xi
\]

This implies that we cannot have \( \lim_{\epsilon \to \tau} \theta(\epsilon) = +\infty \).

Now, let’s consider the remaining possibility that is \( \lim_{\epsilon \to \tau} \theta(\epsilon) = \theta_0(\tau) \). We clearly have \( \lim_{\epsilon \to \tau} \theta'(\epsilon) = +\infty \). Hence,

\[
\exists \xi < \tau, \forall \epsilon \in (\epsilon, \tau), \theta'(\epsilon) \geq \frac{2}{\lambda \sigma^2}
\]

\[
\Rightarrow \forall \epsilon \in (\epsilon, \tau), \theta(\epsilon) \geq \theta(\epsilon) + \frac{2}{\lambda \sigma^2}(\epsilon - \xi)
\]

\[
\Rightarrow \lim_{\epsilon \to \tau} \inf \theta(\epsilon) \geq \frac{\epsilon}{\lambda \sigma^2} + \frac{2}{\lambda \sigma^2}(\xi - \epsilon) > \frac{\tau}{\lambda \sigma^2}
\]

The conclusion is that the maximal interval has no upper bound.

Now, by symmetry the solution is defined on \( \mathbb{R} \). □

One thing remains to be done. In fact, if we have found a function \( \theta(\epsilon) \) that verifies the differential equation and hence a distribution \( m \) coherent with the first order condition, we still need to check that the second order condition is verified to be sure that we characterized a maximum of the optimization criterion. This is the purpose of the following proposition:

**Proposition 5** (Second order condition). Let’s introduce

\[
\Gamma(\epsilon, \theta) = -\lambda^2 \sigma^2 \left(\theta - \frac{\epsilon}{\lambda \sigma^2}\right) \exp\left(-\lambda(1 + r) - \lambda \theta \epsilon + \frac{1}{2} \lambda^2 \theta^2 \sigma^2\right) + \beta m(\theta) C(\epsilon)
\]

Let’s consider the unique function \( \theta(\epsilon) \), given by the preceding proposition, that satisfies \( \forall \epsilon, \Gamma(\epsilon, \theta(\epsilon)) = 0 \) and the conditions of the above proposition.

We have:

\[
\partial_\theta \Gamma(\epsilon, \theta(\epsilon)) < 0
\]
Proof:

First, let’s differentiate the first order condition \( \Gamma(\epsilon, \theta(\epsilon)) = 0 \) with respect to \( \epsilon \). We get:

\[
\partial_\epsilon \Gamma(\epsilon, \theta(\epsilon)) + \theta'(\epsilon) \partial_\theta \Gamma(\epsilon, \theta(\epsilon)) = 0
\]

Thus, the sign of \( \partial_\theta \Gamma(\epsilon, \theta(\epsilon)) \) is the sign of \( -\partial_\epsilon \Gamma(\epsilon, \theta(\epsilon)) \) and we need to prove that \( \partial_\epsilon \Gamma(\epsilon, \theta(\epsilon)) > 0 \).

But:

\[
\partial_\epsilon \Gamma(\epsilon, \theta) = \lambda \exp \left( -\lambda(1 + r) - \lambda \epsilon + \frac{1}{2} \lambda^2 \theta^2 \sigma^2 \right) \left( 1 + \lambda^2 \sigma^2 \theta \left( \theta - \frac{\epsilon}{\lambda \sigma^2} \right) \right) + \beta m(\theta) C'(\epsilon)
\]

This expression is positive for \( \theta = \theta(\epsilon) \) since \( \theta(\epsilon) \geq \frac{\epsilon}{\lambda \sigma^2} \).

This proposition shows that the function \( \theta \) indeed characterizes the behavior of asset managers when they have a tournament-like incentive. In the next section we compute numerically a solution for various parameters in order to quantify the importance of the risk-shifting implied by this tournament effect.

**Numerical examples and discussion**

We will assume that \( f \) is the probability distribution function of a gaussian variable with mean 0 and variance \( s^2 \).

Let us consider the case where \( r = 2\% \), \( \sigma = 20\% \), \( \lambda = 0.5 \) and \( s = 2\% \). We are first considering as a reference case \( \beta = 0.3 \).

For these parameters, the optimal strategy is given by the following figure:

![Figure 1: Optimal strategies for \( \beta = 0.3, \lambda = 0.5, \sigma = 20\%, s = 2\%, r = 2\% \)](image)

The first important feature is that the risk-shifting, defined here as the difference between the optimal strategy \( \theta \) and the Markowitz strategy \( \theta_0 \) is not the same for all asset managers. The risk shifting is in fact maximal for asset managers with intermediate beliefs on the excess return. For instance, the above figure says that an asset manager who believes in an
excess return of (mean) 2% would invest all his money in the risky asset under Markowitz assumption but use a leverage to invest 160% of his wealth in the risky asset in our model with $\beta = 0.3$. Conversely, both the very mild and the very aggressive mutual funds have an optimal strategy that hardly differs from the Markowitz strategy although it is riskier.

An asset manager who believes in a very small excess return does not care much indeed about the ranking since it will always have a rank equal to 50%.

For intermediate asset managers, there is a competition effect that has two components going in the same direction. First, asset managers take more risk in order to improve their ranking at the end of the period. Second, because this is true for every asset manager, each asset manager has to take more risk to avoid being overtaken by his competitors and at the end of the day to maintain his ranking\(^5\). The tournament-induced risk-shifting is then due to an equilibrium effect (this equilibrium being a Nash-MFG one – see [21, 22, 23]). Now, turning to the aggressive asset managers, we can prove\(^6\) that $\lim_{\epsilon \to +\infty} \theta(\epsilon) - \theta_0(\epsilon) = 0$, meaning that the risk-shifting vanishes asymptotically for very large $\epsilon$'s.

To conclude on this numerical example, we saw that the risk-shifting accounted for a large part in the position taken by the intermediate asset managers without any signalling impact since the final ranking of asset managers is the same as in the Markowitz case.

However, one may argue that the value for $\beta$ was chosen rather large and that the ranking effect dominates. To answer this point, consider the case where $\beta$ is doubled and taken equal to 0.6, other parameters remaining unchanged.

The optimal strategies and the associated risk-shifting are respectively given by the following figures:

![Figure 2: Optimal strategies for $\beta = 0.3$ (resp. $\beta = 0.6$), $\lambda = 0.5$, $\sigma = 20\%$, $s = 2\%$, $r = 2\%$](image)

\(^5\)... the ranking being eventually not modified!

\(^6\)... Going back to the function $z = (\theta - \theta_0)^2$ introduced in the proof of proposition 4, we have for $s^2 < \sigma^2$ that:

$$\forall \epsilon > 0, \ 0 \leq z'(\epsilon) + \frac{2}{\lambda \sigma^2} \sqrt{z(\epsilon)} \leq \frac{2\beta e^{\lambda (1+r)} C(\epsilon) f(c) e^{\frac{s^2 x^2}{2\sigma^2}}}{\lambda^2 \sigma^2} \rightarrow_{\epsilon \to +\infty} 0$$

Hence, clearly, $\lim_{\epsilon \to +\infty} z(\epsilon) = 0$
We clearly see that the ranking effect plays a larger role in this case and therefore that the Markowitz part of the criterion played an important role in the $\beta = 0.3$-case. Overall, the investment in the risky asset increases by a large factor and so did the leverage. Consequently, we can easily speak of a tournament-induced risk multiplier that may explain part of the leverage in a market where mutual funds are so numerous that each one has to use every endeavor to distinguish itself from the others.

**Conclusion**

Because they aim at increasing the number of their clients and the money under management, mutual funds incentives are not aligned with the objectives of their current clients. This misalignment due to the flow-performance relationship and its consequences in terms of strategy have been studied and modeled when the inflow is supposed to be a function of the performance relative to a benchmark. Another hypothesis is that the mutual fund industry would be well described by a tournament in which the best performers get the lion’s share of the inflows. We built a model to quantify the risk-shifting induced by the tournament dimension of the mutual fund market and argued that the tournament-induced incentives leads to riskier strategies at equilibrium without any real signalling impact since the ranking of asset managers remains unchanged. Although the literature on tournaments is vast in contract theory and applied to labor market, the impact of the competition between asset managers, if modeled through a tournament-like effect, still deserves research to integrate the different behaviors regarding inflows and outflows (as empirically observed) and to build dynamic models. The mean field game framework seems promising for that purpose.

**References**


