General Intensity Shapes in Optimal Liquidation

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Abstract

We study the optimal liquidation problem using limit orders. If the seminal literature on optimal liquidation, rooted to Almgren-Chriss models, tackles the optimal liquidation problem using a trade-off between market impact and price risk, it only answers the general question of the liquidation rhythm. The very question of the actual way to proceed is indeed rarely dealt with since most classical models use market orders only. Our model, that incorporates both price risk and non-execution risk, answers this question using optimal posting of limit orders. The very general framework we propose regarding the shape of the intensity generalizes both the risk-neutral model presented of Bayraktar and Ludkovski [13] and the model developed in [25], restricted to exponential intensity.

Introduction

Since the late nineties and the first papers on the impact of execution costs on trading strategies (*e.g.* [14]), an important literature has appeared to tackle the problem of optimal liquidation. This literature, often rooted to the seminal papers by Almgren and Chriss [7, 8], has long been characterized by a trade-off between, on one hand, market impact that incites to trade slowly and, on the other hand, price risk that provides incentive to trade fast.

The first family of models, following Almgren and Chriss, considered general instantaneous price impact (sometimes called execution cost) and linear permanent price impact. Several generalizations have been proposed such as an extension to random execution costs [10], or stochastic volatility and stochastic liquidity [5, 6]. Also, many objective functions to design optimal strategies have been proposed and discussed in order to understand the assumptions under which optimal strategies are deterministic (as opposed to adaptive). The initial mean-variance framework has been expressed in the more rigorous framework of a CARA utility function [45], a mean-quadratic-variation framework has been considered [19, 47], initial-time mean-variance criterion has been discussed [9, 40] and the case of a

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general utility function has recently been considered [44] to justify aggressive-in-the-money or passive-in-the-money strategies. Slightly different models have been proposed in this first generation of models (see *e.g.* [30] and [32]) and they all derive from the initial models by Almgren and Chriss since market impact is either permanent or instantaneous and they do not take into account the resilience of the underlying order book.

Another family of models appeared following a paper by Obizhaeva and Wang [41] that directly models the limit order book and considers its resilient dynamic after each trade. This second generation of optimal liquidation models, that incorporate transient market impact, has developed in recent years ([1], [2], [3] and [42]) and raises the theoretical question of the functional form for the transient market impact that are compatible with the absence of price manipulation (see [4], [21] and [23]).

All these models only make use of market orders and hence only consider strategies that consume available liquidity. Hence, they do not consider the possible use of limit orders that provide liquidity to the market nor the post-MIFiD possible use of dark pools. Notwithstanding the preceding criticism, models *à la* Almgren-Chriss provide a rather acceptable answer to the macroscopic question of the optimal rhythm at which liquidation must be proceeded – at least once the instantaneous market impact function has been replaced by an execution cost function modeling the ability to trade over short periods of time with all possible means including limit orders, dark pools and market orders. However, they do not answer the question of the optimal way to proceed in practice and the methods currently used in the industry are seldom based on optimal control models at the microscopic level. This paper provides such a model of optimal liquidation using limit orders[1] and can be used either to liquidate a portfolio as a whole over a few hours or on shorter periods of time to follow a trading curve, be it a TWAP curve, a VWAP curve or an Almgren-Chriss trading curve.

In our approach, a trader sends limit orders to the market (thus providing liquidity instead of consuming it) and does not know when his orders are going to be executed, if at all. As a consequence, the classical trade-off between market impact or execution cost and price risk is not at the core of our model[2]. In our setting, a new risk is in fact borne by the trader because execution is now a random process and this non-execution risk is very different, in its nature, from price risk. This new risk characterizes the recent literature on optimal liquidation, focusing rather on the optimal way to liquidate than on the optimal liquidation rhythm.

The recent literature on optimal liquidation focuses indeed on alternatives to the use of market orders. Kratz and Schoneborn [36] proposed an approach inspired from models of the first family, but with both market orders and access to dark pools. Although they considered no risk-aversion with respect to the new risk borne by the trader, their model is one of the first in this new family of models. The optimal split of large orders across liquidity pools has then been studied by Laruelle, Lehalle and Pagès in [37] but the literature focuses perhaps more today on limit orders than on dark pools. Liquidation with limit orders has indeed been developed by Bayraktar and Ludkovski [13] for general intensity functions but only in a risk-neutral framework. Guéant, Lehalle and Fernandez-Tapia [25] considered in parallel the specific case of an exponential intensity for a risk-adverse agent. Very recently, Huitema [33] considered liquidation involving market orders and limit orders, and Guilbaud and Pham [29] also proposed a liquidation model in a pro-rata microstructure.

[1] and also market orders as it will be discussed in section 5.

[2] Execution costs only appear to liquidate the shares which may be remaining at the end of the period we consider.

One should also note that many models dealing with high-frequency market making have been developed and can be adapted to deal with optimal liquidation. Building on the model proposed by Ho and Stoll [31] and then modified by Avellaneda and Stoikov [11][3], Cartea, Jaimungal and Ricci [18] consider a model with exponential intensity, market impact on the limit order book, adverse selection effects and predictable α. Cartea and Jaimungal [17] recently used a similar model to introduce risk measures for high-frequency trading. Earlier, the same authors proposed a model [16] in which the reference price is modeled by a Hidden Markov Model. Eventually, Guilbaud and Pham [28] also used a model including both market orders and limit orders at best (and next to best) bid and ask together with stochastic spreads.

In this paper, we generalize both [13] and [25]. We indeed consider both general shapes for the intensity functions, and an investor with a CARA utility function. Also, we present a limiting case in which the size of the orders tends to 0 and show that this limiting case approximation is intrinsically linked to the usual continuous framework of Almgren and Chriss although the latter framework only considers market orders.

In the first section, we present the setting of the model and the main hypotheses on execution. The second section is devoted to the resolution of the partial differential equations arising from the control problem. Then, in section 3 we provide illustrations of the model and we exhibit the asymptotic behavior of the quotes, generalizing therefore a result presented in [25]. Section 4 is then dedicated to the study of a limit regime that corresponds to orders of small size. This fourth section leads to results linked to those obtained for the *fluid limit* in [13], here in a risk-adverse setting. This result is exploited in section 5 that draws parallels between our model and the usual Almgren-Chriss framework. We eventually discuss the choice of the intensity and the use of the model in practice.

1 Optimal execution with limit orders: the model

1.1 Setup of the model

Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. We assume that all random variables and stochastic processes are defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

We consider a trader who has to liquidate a portfolio containing a quantity q_0 of a given stock. We suppose that the reference price of the stock (that can be considered the first bid quote for example) follows a Brownian motion[4]:

$$dS_t = \sigma dW_t$$

The trader under consideration will continuously propose an ask quote denoted S_t^a and will hence sell shares according to the rate of arrival of liquidity-consuming orders at the prices he quotes.

His inventory q, that is the quantity he holds, is given by $q_t = q_0 - \Delta N_t$ where N is a point process giving the number of executed orders, each order being of size Δ and entirely

[3]See also [26].

[4]The model can be generalized with the adjunction of a drift.

filled[5] – we suppose obviously that Δ is a fraction of q_0. Arrival rates depend on the (ask) price S_t^a quoted by the trader at date t and we posit that the intensity λ associated to N is of the following form, that depends on Δ:

$$\lambda(s^a, s, \Delta) = \Lambda_\Delta(s^a - s)$$

where $\Lambda_\Delta : \mathbb{R} \to \mathbb{R}_+$ satisfies the following assumptions[6]:

- Λ_Δ is strictly decreasing – the cheaper the order price, the faster it will be executed,
- $\lim_{\delta \to +\infty} \Lambda_\Delta(\delta) = 0$,
- $\Lambda_\Delta \in C^2$,
- $\Lambda_\Delta(\delta)\Lambda''_\Delta(\delta) \leq 2\Lambda'_\Delta(\delta)^2$.

As a consequence of his trades, the trader has an amount of cash whose dynamics is given by:

$$dX_t = S_t^a \Delta dN_t.$$

Now, coming to the liquidation problem, the trader has a time horizon T to liquidate his shares and his goal is to optimize the expected utility of his P&L at time T. We focus on CARA utility functions so that the trader maximizes:

$$\mathbb{E}\left[-\exp\left(-\gamma\left(X_T + q_T(S_T - \ell(q_T))\right)\right)\right]$$

where $\gamma > 0$ is the absolute risk aversion parameter characterizing the trader, where X_T is the amount of cash at time T and where q_T is the remaining quantity of shares in the inventory at time T. In this setting, the trader can sell the shares remaining at time T in his portfolio at a price below the reference price, namely $S_T - \ell(q_T)$, the function ℓ being a positive and increasing function measuring execution cost.

1.2 From Hamilton-Jacobi to a system of ODEs

The optimization problem set up in the preceding section can be solved using classical Bellman tools. To this purpose, we introduce the Hamilton-Jacobi-Bellman equation associated to the optimization problem, where the unknown u is going to be the value function of the problem:

$$\text{(HJB)} \qquad 0 = \partial_t u_\Delta(t,x,q,s) + \frac{1}{2}\sigma^2 \partial^2_{ss} u_\Delta(t,x,q,s)$$
$$+ \sup_{s^a} \Lambda_\Delta(s^a - s)\left[u_\Delta(t, x + \Delta s^a, q - \Delta, s) - u_\Delta(t,x,q,s)\right]$$

with the final condition:

$$u_\Delta(T,x,q,s) = -\exp\left(-\gamma\left(x + q(s - \ell(q))\right)\right)$$

and the boundary condition:

$$u_\Delta(t,x,0,s) = -\exp\left(-\gamma x\right)$$

[5]There is no partial fill in this model.

[6]We assume that Λ_Δ is defined on the entire real line and not only on $\mathbb{R}_+$. We shall discuss in section 5 the interpretation of quotes below the reference price – hereafter called negative quotes.

Since we use a CARA function, we can factor out the Mark-to-Market (MtM) value of the portfolio and we consider the change of variables $u_\Delta(t,x,q,s) = -\exp\left(-\gamma(x+qs+\theta_\Delta(t,q))\right)$. In that case, the above HJB equation with 4 variables is (formally) reduced to the following system of ODEs indexed by q:

$$(\mathrm{HJ}_{\theta_\Delta}) \qquad 0 = \gamma\partial_t\theta_\Delta(t,q) - \frac{1}{2}\gamma^2\sigma^2q^2 + H_\Delta\left(\frac{\theta_\Delta(t,q)-\theta_\Delta(t,q-\Delta)}{\Delta}\right)$$

with

$$\theta_\Delta(T,q) = -\ell(q)q, \qquad \theta_\Delta(t,0) = 0$$

where

$$H_\Delta(p) = \sup_\delta \Lambda_\Delta(\delta)\left(1-e^{-\gamma\Delta(\delta-p)}\right)$$

2 Solving the optimal control problem

This section aims at solving the optimal control problem set up in the preceding section. We first concentrate on the equation $(\mathrm{HJ}_{\theta_\Delta})$ and we then provide a verification theorem that indeed provides a solution to the control problem and characterizes in a simple way the optimal quotes.

2.1 A solution to $(\mathrm{HJ}_{\theta_\Delta})$

We start with a lemma about the hamiltonian function H_Δ.

Lemma 1. *Let us define* $L_\Delta(p,\delta) = \Lambda_\Delta(\delta)\left(1-e^{-\gamma\Delta(\delta-p)}\right)$.

$\forall p\in\mathbb{R}$, $\delta\mapsto L_\Delta(p,\delta)$ reaches its maximum at $\tilde\delta^*_\Delta(p)$ uniquely characterized by:

$$\tilde\delta^*_\Delta(p) - \frac{1}{\gamma\Delta}\log\left(1-\gamma\Delta\frac{\Lambda_\Delta(\tilde\delta^*_\Delta(p))}{\Lambda'_\Delta(\tilde\delta^*_\Delta(p))}\right) = p$$

Moreover, $p\mapsto\tilde\delta^*_\Delta(p)$ is a C^1 function.

Subsequently H_Δ is a C^1 function with:

$$H_\Delta(p) = \gamma\Delta\frac{\Lambda_\Delta(\tilde\delta^*_\Delta(p))^2}{\gamma\Delta\Lambda_\Delta(\tilde\delta^*_\Delta(p)) - \Lambda'_\Delta(\tilde\delta^*_\Delta(p))}$$

Proof:

First, let us notice that $L_\Delta(p,p)=0$ and that $\lim_{\delta\to+\infty}L_\Delta(p,\delta)=0$.

Now, if we differentiate, we get:

$$\partial_\delta L_\Delta(p,\delta) = \Lambda'_\Delta(\delta)\left(1-e^{-\gamma\Delta(\delta-p)}\right) + \gamma\Delta\Lambda_\Delta(\delta)e^{-\gamma\Delta(\delta-p)}$$

Hence, $\partial_\delta L_\Delta(p,p) = \gamma\Delta\Lambda_\Delta(p) > 0$ and there is at least one $\delta^*\in(p,+\infty)$ such that $\partial_\delta L_\Delta(p,\delta^*)=0$ and corresponding to a maximum of $L_\Delta(p,\cdot)$.

Such a δ^* must satisfy:

$$\delta^* - \frac{1}{\gamma\Delta}\log\left(1-\gamma\Delta\frac{\Lambda_\Delta(\delta^*)}{\Lambda'_\Delta(\delta^*)}\right) = p$$

Now $f(x) = x - \frac{1}{\gamma\Delta}\log\left(1 - \gamma\Delta\frac{\Lambda_\Delta(x)}{\Lambda'_\Delta(x)}\right)$ defines a strictly increasing function since

$$\begin{aligned} f'(x) &= 1 + \frac{\left(\frac{\Lambda_\Delta(x)}{\Lambda'_\Delta(x)}\right)'}{1 - \gamma\Delta\frac{\Lambda_\Delta(x)}{\Lambda'_\Delta(x)}} \\ &= 1 + \frac{\Lambda'_\Delta(x)^2 - \Lambda_\Delta(x)\Lambda''_\Delta(x)}{\Lambda'_\Delta(x)^2 - \gamma\Delta\Lambda_\Delta(x)\Lambda'_\Delta(x)} \\ &= \frac{-\gamma\Delta\Lambda_\Delta(x)\Lambda'_\Delta(x)}{\Lambda'_\Delta(x)^2 - \gamma\Delta\Lambda_\Delta(x)\Lambda'_\Delta(x)} + \frac{2\Lambda'_\Delta(x)^2 - \Lambda_\Delta(x)\Lambda''_\Delta(x)}{\Lambda'_\Delta(x)^2 - \gamma\Delta\Lambda_\Delta(x)\Lambda'_\Delta(x)} \end{aligned}$$

and this expression is strictly positive because of the hypotheses on Λ_Δ.

Hence $L_\Delta(p,\cdot)$ reaches its unique maximum at $\tilde{\delta}^*_\Delta(p)$ uniquely characterized by:

$$\tilde{\delta}^*_\Delta(p) - \frac{1}{\gamma\Delta}\log\left(1 - \gamma\Delta\frac{\Lambda_\Delta(\tilde{\delta}^*_\Delta(p))}{\Lambda'_\Delta(\tilde{\delta}^*_\Delta(p))}\right) = p$$

Using the implicit function theorem, this also gives that $p \mapsto \tilde{\delta}^*_\Delta(p)$ is a C^1 function.

Plugging the relation for $\tilde{\delta}^*_\Delta(p)$ in the definition of H_Δ then gives the last part of the lemma. □

Now, we are going to prove a comparison principle for the system of ODEs. This result is useful in two ways. First, it gives *a priori* bounds that will allow us to prove the existence of a solution to ($\mathrm{HJ}_{\theta_\Delta}$). Second, it will provide bounds to θ_Δ along with Lipschitz-type regularity properties, both being independent of Δ.

Proposition 1 (Comparison principle)**.** *Let $\tau \in [0,T)$, $k \in \{0,1,\dots,\frac{q_0}{\Delta}\}$.*

Let $\underline{\theta}_\Delta : [\tau,T]\times\{0,\Delta,\dots,k\Delta\} \to \mathbb{R}$ be a C^1 function with respect to time with

$$\forall q \in \{0,\Delta,\dots,k\Delta\},\quad \underline{\theta}_\Delta(T,q) \le -\ell(q)q \qquad \forall t\in[\tau,T],\quad \underline{\theta}_\Delta(t,0)\le 0$$

and

$$0 \le \gamma\partial_t\underline{\theta}_\Delta(t,q) - \frac{1}{2}\gamma^2\sigma^2q^2 + H_\Delta\left(\frac{\underline{\theta}_\Delta(t,q) - \underline{\theta}_\Delta(t,q-\Delta)}{\Delta}\right)$$

Let $\overline{\theta}_\Delta : [\tau,T]\times\{0,\Delta,\dots,k\Delta\} \to \mathbb{R}$ be a C^1 function with respect to time with

$$\forall q \in \{0,\Delta,\dots,k\Delta\},\quad \overline{\theta}_\Delta(T,q) \ge -\ell(q)q \qquad \forall t\in[\tau,T],\quad \overline{\theta}_\Delta(t,0)\ge 0$$

and

$$0 \ge \gamma\partial_t\overline{\theta}_\Delta(t,q) - \frac{1}{2}\gamma^2\sigma^2q^2 + H_\Delta\left(\frac{\overline{\theta}_\Delta(t,q) - \overline{\theta}_\Delta(t,q-\Delta)}{\Delta}\right)$$

Then

$$\overline{\theta}_\Delta \ge \underline{\theta}_\Delta$$

Proof:

Let $\alpha > 0$.

Let us consider a point (t^*_α, q^*_α) such that

$$\underline{\theta}_\Delta(t^*_\alpha, q^*_\alpha) - \overline{\theta}_\Delta(t^*_\alpha, q^*_\alpha) - \alpha(T - t^*_\alpha) = \sup_{(t,q)\in[\tau,T]\times\{0,\dots,k\Delta\}} \underline{\theta}_\Delta(t,q) - \overline{\theta}_\Delta(t,q) - \alpha(T-t)$$

If $t^*_\alpha \neq T$ and $q^*_\alpha \neq 0$ then:

$$\partial_t \underline{\theta}_\Delta(t^*_\alpha, q^*_\alpha) - \partial_t \overline{\theta}_\Delta(t^*_\alpha, q^*_\alpha) \leq -\alpha$$

Now, by definition of the functions $\underline{\theta}_\Delta$ and $\overline{\theta}_\Delta$

$$0 \leq \gamma \left[\partial_t \underline{\theta}_\Delta(t^*_\alpha, q^*_\alpha) - \partial_t \overline{\theta}_\Delta(t^*_\alpha, q^*_\alpha)\right]$$

$$+ \left[H_\Delta\left(\frac{\underline{\theta}_\Delta(t^*_\alpha, q^*_\alpha) - \underline{\theta}_\Delta(t^*_\alpha, q^*_\alpha - \Delta)}{\Delta}\right) - H_\Delta\left(\frac{\overline{\theta}_\Delta(t^*_\alpha, q^*_\alpha) - \overline{\theta}_\Delta(t^*_\alpha, q^*_\alpha - \Delta)}{\Delta}\right)\right]$$

Now, by definition of (t^*_α, q^*_α), since $q^*_\alpha \neq 0$:

$$\underline{\theta}_\Delta(t^*_\alpha, q^*_\alpha) - \overline{\theta}_\Delta(t^*_\alpha, q^*_\alpha) \geq \underline{\theta}_\Delta(t^*_\alpha, q^*_\alpha - \Delta) - \overline{\theta}_\Delta(t^*_\alpha, q^*_\alpha - \Delta)$$

$$\underline{\theta}_\Delta(t^*_\alpha, q^*_\alpha) - \underline{\theta}_\Delta(t^*_\alpha, q^*_\alpha - \Delta) \geq \overline{\theta}_\Delta(t^*_\alpha, q^*_\alpha - \Delta) - \overline{\theta}_\Delta(t^*_\alpha, q^*_\alpha)$$

Since H_Δ is a decreasing function, we have

$$H_\Delta\left(\frac{\underline{\theta}_\Delta(t^*_\alpha, q^*_\alpha) - \underline{\theta}_\Delta(t^*_\alpha, q^*_\alpha - \Delta)}{\Delta}\right) \leq H_\Delta\left(\frac{\overline{\theta}_\Delta(t^*_\alpha, q^*_\alpha) - \overline{\theta}_\Delta(t^*_\alpha, q^*_\alpha - \Delta)}{\Delta}\right)$$

This leads to $0 \leq -\gamma\alpha$ which is not possible.

Therefore $t^*_\alpha = T$ or $q^*_\alpha = 0$ so that:

$$\sup_{(t,q)\in[\tau,T]\times\{0,\dots,k\Delta\}} \underline{\theta}_\Delta(t,q) - \overline{\theta}_\Delta(t,q) - \alpha(T-t)$$

$$= \max\left(\sup_{q\in\{0,\dots,k\Delta\}} \underline{\theta}_\Delta(T,q) - \overline{\theta}_\Delta(T,q), \sup_{t\in[\tau,T]} \underline{\theta}_\Delta(t,0) - \overline{\theta}_\Delta(t,0) - \alpha(T-t)\right) \leq 0$$

Thus $\forall (t,q) \qquad \underline{\theta}_\Delta(t,q) - \overline{\theta}_\Delta(t,q) \leq \alpha T$ and sending α to 0 proves our result. □

We are now ready to solve the equation $(\mathrm{HJ}_{\theta_\Delta})$

Proposition 2. *There exists a unique function $\theta_\Delta : [0,T] \times \{0, \Delta, 2\Delta, \dots, q_0\} \to \mathbb{R}$, C^1 in time, solution of* $(\mathrm{HJ}_{\theta_\Delta})$.

Proof:

We proceed by induction on q. For $q = 0$, we have by definition $\theta_\Delta(t, 0) = 0$.

Now, for a given $q \in \{\Delta, 2\Delta, \dots, q_0\}$, let us suppose that $\theta_\Delta(\cdot, q') : [0,T] \to \mathbb{R}$ is a C^1 function $\forall q' \leq q - \Delta$. Then, the ODE

$$0 = \gamma \partial_t \theta_\Delta(t,q) - \frac{1}{2}\gamma^2\sigma^2 q^2 + H_\Delta\left(\frac{\theta_\Delta(t,q) - \theta_\Delta(t, q-\Delta)}{\Delta}\right)$$

with the terminal condition

$$\theta_\Delta(T,q) = -\ell(q)q$$

satisfies the assumptions of Cauchy-Lipschitz theorem. Consequently, there exists a solution $t \mapsto \theta_\Delta(t,q)$ on a maximal interval that is a sub-interval of $[0,T]$ and we want to show that this sub-interval is $[0,T]$ itself.

To prove that, let suppose by contradiction that $(\tilde{t},T]$ is the maximal interval with $\tilde{t}>0$. Let us notice that, because H_Δ is positive, $t \mapsto \theta_\Delta(t,q)+\frac{1}{2}\gamma\sigma^2 q^2(T-t)$ is decreasing. Hence, the only possibility for $(\tilde{t},T]$ to be a maximal interval in $[0,T]$ is that $\lim_{t\to\tilde{t}^+}\theta_\Delta(t,q)=+\infty$.

But, if we consider $\eta>0$, $k=\frac{q}{\Delta}$ and $\tau=\tilde{t}+\eta$, the two functions $\underline{\theta}_\Delta=\theta_\Delta$ and $\overline{\theta}_\Delta(t,q)=\frac{1}{\gamma}H_\Delta(0)(T-t)$ satisfy the assumptions of the above comparison principle.
Hence, $\forall\eta>0, \forall t\in[\tilde{t}+\eta,T], \theta_\Delta(t,q)\leq\frac{1}{\gamma}H_\Delta(0)(T-\tilde{t})$, in contradiction with the fact that $\lim_{t\to\tilde{t}^+}\theta_\Delta(t,q)=+\infty$.

Hence, $t\mapsto\theta_\Delta(t,q)$ is defined on $[0,T]$ and this proves the result. □

2.2 Verification theorem and optimal quotes

Now, we can solve the initial optimal control problem and find the optimal quotes at which the trader should send his limit orders.

Theorem 1 (Verification theorem and optimal quotes)**.** *Let us consider the solution θ_Δ of the system* ($\mathrm{HJ}_{\theta_\Delta}$).
Then, $u_\Delta(t,x,q,s)=-\exp(-\gamma(x+qs+\theta_\Delta(t,q)))$ defines a solution to (HJB) and is the value function of the optimal control problem, i.e.:

$$u_\Delta(t,x,q,s)=\sup_{\delta\in\mathcal{A}(t)}\mathbb{E}\left[-\exp\left(-\gamma\left(X_T^{t,x,\delta}+q_T^{t,q,\delta}(S_T^{t,s}-\ell(q_T^{t,q,\delta}))\right)\right)\right]$$

where $\mathcal{A}(t)$ is the set of predictable processes on $[t,T]$, bounded from below and where:

$$dS_\tau^{t,s}=\sigma dW_\tau,\qquad S_t^{t,s}=s$$

$$dX_\tau^{t,x,\delta}=(S_\tau+\delta_\tau)\Delta dN_\tau,\qquad X_t^{t,x,\delta}=x$$

$$dq_\tau^{t,q,\delta}=-\Delta dN_\tau,\qquad q_t^{t,q,\delta}=q$$

where the point process has stochastic intensity $(\lambda_\tau)_\tau$ with $\lambda_\tau=\Lambda_\Delta(\delta_\tau)1_{q_{\tau-}\geq\Delta}$.

Moreover the optimal ask quote $S_t^a=S_t+(\delta_\Delta^*)_t$, for $q_t>0$, is characterized by:

$$(\delta_\Delta^*)_t=\tilde{\delta}_\Delta^*\left(\frac{\theta_\Delta(t,q_t)-\theta_\Delta(t,q_t-\Delta)}{\Delta}\right)$$

where $\tilde{\delta}_\Delta^*(p)$ is uniquely characterized, as in Lemma 1, by:

$$\tilde{\delta}_\Delta^*(p)-\frac{1}{\gamma\Delta}\log\left(1-\gamma\Delta\frac{\Lambda_\Delta(\tilde{\delta}_\Delta^*(p))}{\Lambda'_\Delta(\tilde{\delta}_\Delta^*(p))}\right)=p$$

Proof:

From the very definition of θ_Δ and u_Δ it is straightforward to see that u_Δ is a solution of (HJB).

We indeed have that the boundary condition and the terminal condition are satisfied. For $q \geq \Delta$ we have:

$$\begin{aligned}
&\partial_t u_\Delta(t,x,q,s) + \frac{1}{2}\sigma^2 \partial^2_{ss} u_\Delta(t,x,q,s) \\
&+ \sup_\delta \Lambda_\Delta(\delta)\left[u_\Delta(t,x+\Delta s+\Delta\delta, q-\Delta, s) - u_\Delta(t,x,q,s)\right] \\
&= -\gamma u_\Delta(t,x,q,s)\partial_t\theta_\Delta(t,q) + \frac{1}{2}\sigma^2\gamma^2 q^2 u_\Delta(t,x,q,s) \\
&+ \sup_\delta \Lambda_\Delta(\delta) u_\Delta(t,x,q,s)\left(\exp\left(-\gamma(\Delta\delta - (\theta_\Delta(t,q) - \theta_\Delta(t,q-\Delta)))\right) - 1\right) \\
&= -u_\Delta(t,x,q,s)\Bigg[\gamma\partial_t\theta_\Delta(t,q) - \frac{1}{2}\sigma^2\gamma^2 q^2 \\
&+ \sup_\delta \Lambda_\Delta(\delta)\left(1 - \exp\left(-\gamma(\Delta\delta - (\theta_\Delta(t,q) - \theta_\Delta(t,q-\Delta)))\right)\right)\Bigg] = 0 \\
&= -u_\Delta(t,x,q,s)\left[\gamma\partial_t\theta_\Delta(t,q) - \frac{1}{2}\sigma^2\gamma^2 q^2 + H_\Delta\left(\frac{\theta_\Delta(t,q) - \theta_\Delta(t,q-\Delta)}{\Delta}\right)\right] = 0
\end{aligned}$$

Now, we need to verify that u_Δ is indeed the value function associated to the problem and to prove that our candidate $(\delta^*_\Delta)_t$ is indeed the optimal control. To that purpose, let us consider a control $\delta \in \mathcal{A}(t)$ and let us consider the following processes for $\tau \in [t,T]$:

$$\begin{aligned}
dS^{t,s}_\tau &= \sigma dW_\tau, \qquad S^{t,s}_t = s \\
dX^{t,x,\delta}_\tau &= (S_\tau + \delta_\tau)\Delta dN_\tau, \qquad X^{t,x,\delta}_t = x \\
dq^{t,q,\delta}_\tau &= -\Delta dN_\tau, \qquad q^{t,q,\delta}_t = q
\end{aligned}$$

where the point process has stochastic intensity $(\lambda_\tau)_\tau$ with $\lambda_\tau = \Lambda_\Delta(\delta_\tau)1_{q_{\tau-}\geq\Delta}$[7].

Now, let us write Itô's formula for u_Δ[8]:

$$\begin{aligned}
&u_\Delta(T, X^{t,x,\delta}_{T-}, q^{t,q,\delta}_{T-}, S^{t,s}_T) = u_\Delta(t,x,q,s) \\
&+ \int_t^T \left(\partial_\tau u_\Delta(\tau, X^{t,x,\delta}_{\tau-}, q^{t,q,\delta}_{\tau-}, S^{t,s}_\tau) + \frac{\sigma^2}{2}\partial^2_{ss} u_\Delta(\tau, X^{t,x,\delta}_{\tau-}, q^{t,q,\delta}_{\tau-}, S^{t,s}_\tau)\right) d\tau \\
&+ \int_t^T \left(u_\Delta(\tau, X^{t,x,\delta}_{\tau-} + \Delta S^{t,s}_\tau + \Delta\delta_\tau, q^{t,q,\delta}_{\tau-} - \Delta, S^{t,s}_\tau) - u_\Delta(\tau, X^{t,x,\delta}_{\tau-}, q^{t,q,\delta}_{\tau-}, S^{t,s}_\tau)\right)\lambda_\tau d\tau \\
&+ \int_t^T \sigma\partial_s u_\Delta(\tau, X^{t,x,\delta}_{\tau-}, q^{t,q,\delta}_{\tau-}, S^{t,s}_\tau) dW_\tau \\
&+ \int_t^T \left(u_\Delta(\tau, X^{t,x,\delta}_{\tau-} + \Delta S^{t,s}_\tau + \Delta\delta_\tau, q^{t,q,\delta}_{\tau-} - \Delta, S^{t,s}_\tau) - u_\Delta(\tau, X^{t,x,\delta}_{\tau-}, q^{t,q,\delta}_{\tau-}, S^{t,s}_\tau)\right) dM_\tau
\end{aligned}$$

where M is the compensated process associated to N for the intensity process $(\lambda_\tau)_\tau$.

Now, we have to ensure that the last two integrals consist of martingales so that their mean is 0. To that purpose, let us notice that $\partial_s u = -\gamma q u$ and hence, since the process $q^{t,q,\delta}$ takes values between 0 and q, we just have to prove that:

[7] This intensity being bounded since δ is bounded from below.

[8] The equality is still valid when $q_\tau = 0$ because of the boundary condition for u_Δ and because the intensity process is then assumed to be 0.

$$\mathbb{E}\left[\int_t^T u_\Delta(\tau, X_{\tau-}^{t,x,\delta}, q_{\tau-}^{t,q,\delta}, S_\tau^{t,s})^2 d\tau\right] < +\infty$$

$$\mathbb{E}\left[\int_t^T \left|u_\Delta(\tau, X_{\tau-}^{t,x,\delta} + \Delta S_\tau^{t,s} + \Delta\delta_\tau, q_{\tau-}^{t,q,\delta} - \Delta, S_\tau^{t,s})\right| \lambda_\tau d\tau\right] < +\infty$$

and

$$\mathbb{E}\left[\int_t^T \left|u_\Delta(\tau, X_{\tau-}^{t,x,\delta}, q_{\tau-}^{t,q,\delta}, S_\tau^{t,s})\right| \lambda_\tau d\tau\right] < +\infty$$

We have:

$$u_\Delta(\tau, X_\tau^{t,x,\delta}, q_\tau^{t,q,\delta}, S_\tau^{t,s})^2 \leq \exp\left(2\gamma\|\theta_\Delta\|_\infty\right)\exp\left(-2\gamma(X_\tau^{t,x,\delta} + q_\tau^{t,q,\delta} S_\tau^{t,s})\right)$$

$$\leq \exp\left(2\gamma\|\theta_\Delta\|_\infty\right)\exp\left(-2\gamma(x - q\|\delta^-\|_\infty + 2q \inf_{\tau\in[t,T]} S_\tau^{t,s} 1_{\inf_{\tau\in[t,T]} S_\tau^{t,s}<0})\right)$$

$$\leq \exp\left(2\gamma\|\theta_\Delta\|_\infty\right)\exp\left(-2\gamma(x - q\|\delta^-\|_\infty)\right)\left(1 + \exp\left(-2\gamma q \inf_{\tau\in[t,T]} S_\tau^{t,s}\right)\right)$$

Hence:

$$\mathbb{E}\left[\int_t^T u_\Delta(\tau, X_\tau^{t,x,\delta}, q_\tau^{t,q,\delta}, S_\tau^{t,s})^2 d\tau\right] = \mathbb{E}\left[\int_t^T u_\Delta(\tau, X_{\tau-}^{t,x,\delta}, q_{\tau-}^{t,q,\delta}, S_\tau^{t,s})^2 d\tau\right]$$

$$\leq \exp\left(2\gamma\|\theta_\Delta\|_\infty\right)\exp\left(-2\gamma(x - q\|\delta^-\|_\infty)\right)(T-t)\left(1 + \mathbb{E}\left[\exp\left(-2\gamma q \inf_{\tau\in[t,T]} S_\tau^{t,s}\right)\right]\right) < +\infty$$

because of the law of $\inf_{\tau\in[t,T]} S_\tau^{t,s}$.

Now, the same argument works for the second and third integrals, noticing that δ is bounded from below and that λ is bounded.

Hence, since we have, by construction[9]

$$\partial_\tau u_\Delta(\tau, X_{\tau-}^{t,x,\delta}, q_{\tau-}^{t,q,\delta}, S_\tau^{t,s}) + \frac{\sigma^2}{2}\partial_{ss}^2 u_\Delta(\tau, X_{\tau-}^{t,x,\delta}, q_{\tau-}^{t,q,\delta}, S_\tau^{t,s})$$

$$+\left(u_\Delta(\tau, X_{\tau-}^{t,x,\delta} + \Delta S_\tau^{t,s} + \Delta\delta_t, q_{\tau-}^{t,q,\delta} - \Delta, S_\tau^{t,s}) - u_\Delta(\tau, X_{\tau-}^{t,x,\delta}, q_{\tau-}^{t,q,\delta}, S_\tau^{t,s})\right)\lambda_\tau \leq 0$$

we obtain that

$$\mathbb{E}\left[u_\Delta(T, X_T^{t,x,\delta}, q_T^{t,q,\delta}, S_T^{t,s})\right] = \mathbb{E}\left[u_\Delta(T, X_{T-}^{t,x,\delta}, q_{T-}^{t,q,\delta}, S_T^{t,s})\right] \leq u_\Delta(t,x,q,s)$$

and this is true for all $\delta \in \mathcal{A}(t)$. Since for $\delta_t = (\delta_\Delta^*)_t$ we have an equality in the above inequality by construction of the function $\tilde{\delta}_\Delta^*$, we obtain that:

$$\sup_{\delta\in\mathcal{A}(t)} \mathbb{E}\left[u_\Delta(T, X_T^{t,x,\delta}, q_T^{t,q,\delta}, S_T^{t,s})\right] \leq u_\Delta(t,x,q,s) = \mathbb{E}\left[u_\Delta(T, X_T^{t,x,\delta_\Delta^*}, q_T^{t,q,\delta_\Delta^*}, S_T^{t,s})\right]$$

i.e.

$$\sup_{\delta\in\mathcal{A}(t)} \mathbb{E}\left[-\exp\left(-\gamma\left(X_T^{t,x,\delta} + q_T^{t,q,\delta}(S_T^{t,s} - \ell(q_T^{t,q,\delta}))\right)\right)\right]$$

[9]This inequality is also true when the portfolio is empty because of the boundary conditions.

$$\leq u_\Delta(t,x,q,s) = \mathbb{E}\left[-\exp\left(-\gamma\left(X_T^{t,x,\delta^*_\Delta} + q_T^{t,q,\delta^*_\Delta}(S_T^{t,s} - \ell(q_T^{t,q,\delta^*_\Delta}))\right)\right)\right]$$

This proves that u is the value function and that $t \mapsto (\delta^*_\Delta)_t$ is optimal. □

Theorem 1 provides a simple way to compute the optimal quotes. One has indeed to solve the triangular system of ODEs ($\mathrm{HJ}_{\theta_\Delta}$) to obtain the function θ_Δ. Numerically, this does not constitute any difficulty. Then, once θ_Δ has been computed, the optimal quotes are given by the simple expression $\tilde{\delta}^*_\Delta\left(\frac{\theta_\Delta(t,q_t)-\theta_\Delta(t,q_t-\Delta)}{\Delta}\right)$ where the function $\tilde{\delta}^*_\Delta$ is implicitly characterized by

$$\tilde{\delta}^*_\Delta(p) - \frac{1}{\gamma\Delta}\log\left(1 - \gamma\Delta\frac{\Lambda_\Delta(\tilde{\delta}^*_\Delta(p))}{\Lambda'_\Delta(\tilde{\delta}^*_\Delta(p))}\right) = p$$

and can then be easily computed using Newton's method for instance.

To better understand the model, we are now going to relate our results with those of [25], provide asymptotic results about the quotes and discuss the role of the parameters.

3 Examples and properties

In the above part, we generalized a model already used in [25], in which the intensity functions had exponential shape[10]: $\Lambda_\Delta(\delta) = A_\Delta e^{-k_\Delta\delta}$.

In the case of exponential intensity, we can write the results of [25] (in a slightly more general case than in the original paper) using the language of this paper. In fact, the reason why closed-form solutions can be obtained in the exponential case is because the equation ($\mathrm{HJ}_{\theta_\Delta}$) simplifies to a linear system of equations once we replace the unknown θ_Δ by $\exp\left(\frac{k_\Delta}{\Delta}\theta_\Delta\right)$:

Proposition 3. *Let us consider $\theta_\Delta : [0,T]\times\{0,\Delta,2\Delta,\dots,q_0\} \to \mathbb{R}$, C^1 in time, solution of ($\mathrm{HJ}_{\theta_\Delta}$) when $\Lambda_\Delta(\delta) = A_\Delta e^{-k_\Delta\delta}$.*
Let us define $w_\Delta = \exp\left(\frac{k_\Delta}{\Delta}\theta_\Delta\right)$. Then, w_Δ solves:

$$\partial_t w_\Delta(t,q) = \frac{1}{2\Delta}\gamma k_\Delta\sigma^2 q^2 w_\Delta(t,q) - A_\Delta\left(1+\frac{\gamma\Delta}{k_\Delta}\right)^{-1-\frac{k_\Delta}{\gamma\Delta}} w_\Delta(t,q-\Delta)$$

with

$$w_\Delta(T,q) = e^{-\frac{k_\Delta}{\Delta}\ell(q)q}, \qquad w_\Delta(t,0) = 1$$

and the optimal quote, for $q_t > 0$, is given by:

$$(\delta^*_\Delta)_t = \frac{1}{k_\Delta}\log\left(\frac{w_\Delta(t,q_t)}{w_\Delta(t,q_t-\Delta)}\right) + \frac{1}{\gamma\Delta}\log\left(1+\frac{\gamma\Delta}{k_\Delta}\right)$$

We provide below numerical approximations of both the function $\theta_\Delta(t,q)$ and the optimal control function $\delta^*_\Delta(t,q)$ that will motivate general results. The first result presented in this section concerns the behavior of the quotes as the time horizon T increases. It generalizes

[10] Another example has been considered in the literature on optimal execution, namely power-law intensity functions in the risk-neutral case (see [13]). These functions are only defined for positive δ and do not enter the framework developed above, although this framework can be adapted to take these intensity functions into account.

an asymptotic result obtained for exponential intensity functions in [25]. The second one, presented in the next section, concerns the limit regime as the size Δ of orders goes to 0.

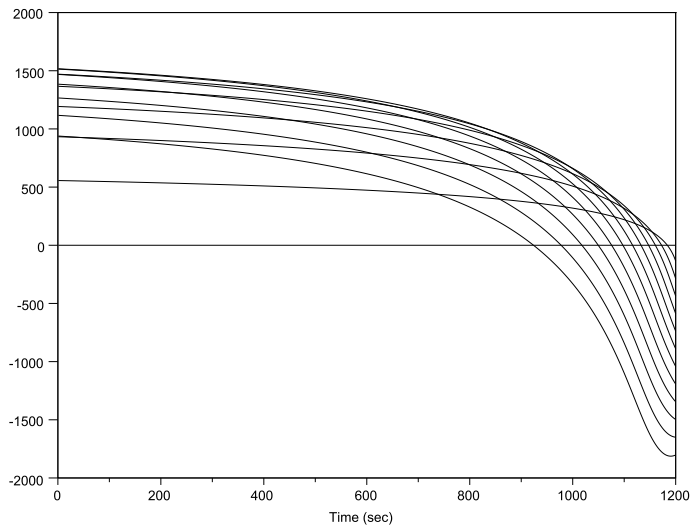

Figure 1: Solution $\theta_\Delta(t,q)$ for $\Lambda_\Delta(\delta) = Ae^{-k\delta}$, $q_0 = 600$, $\Delta = 50$, $T = 1200$ (s), $\sigma = 0.3$ (Tick.s$^{-\frac{1}{2}}$), $A = 0.1$ (s^{-1}), $k = 0.3$ (Tick^{-1}), $\gamma = 0.001$ (Tick^{-1}) and $\ell = 3$ (Tick). The index $q \in \{0, 50, \ldots, 600\}$ of each curve can be read from the terminal values.

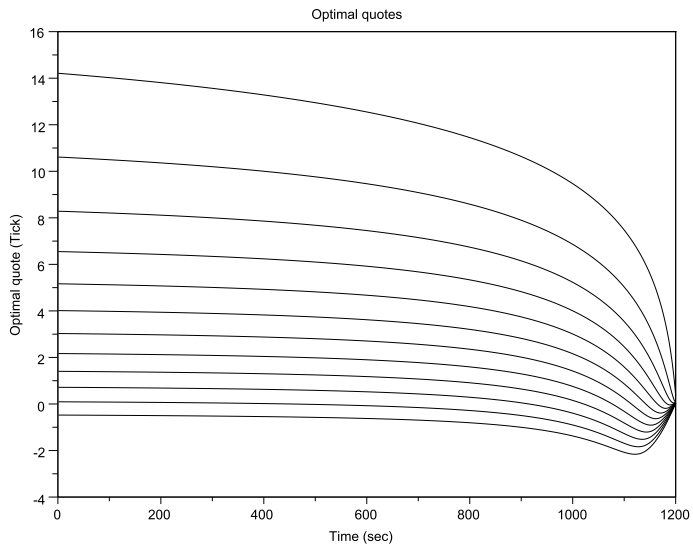

Figure 2: Optimal control $\delta^*_\Delta(t,q)$ for $\Lambda_\Delta(\delta) = Ae^{-k\delta}$, $q_0 = 600$, $\Delta = 50$, $T = 1200$ (s), $\sigma = 0.3$ (Tick.s$^{-\frac{1}{2}}$), $A = 0.1$ (s^{-1}), $k = 0.3$ (Tick^{-1}), $\gamma = 0.001$ (Tick^{-1}) and $\ell = 3$ (Tick). The lower the quote the larger the inventory $q \in \{50, 100, \ldots, 600\}$.

Figure 1 and Figure 2 represent respectively the solution θ_Δ and the optimal quotes

$\delta^*_\Delta(t,q)$ as given by Theorem 1[11]. We see on Figure 1 that θ_Δ is not a monotonic function of q. This means that the certainty equivalent of holding a quantity q of shares, once the MtM value of these shares has been removed is not a monotonic function of q. At the time horizon T, the function is decreasing but far from T two effects are at stake. On one hand, when there are more shares in the portfolio, there will be more trades and hence more opportunities to make money through limit orders: this goes in the direction of an increasing function $\theta_\Delta(t,\cdot)$. On the other hand, the larger the inventory to liquidate, the more price risk, and this goes in the direction of a decreasing function $\theta_\Delta(t,\cdot)$.

In spite of this absence of monotonicity, Figure 1 and Figure 2 suggest to study the asymptotic case $T \to \infty$ and the result obtained in [25] indeed generalizes to nearly any intensity function:

Proposition 4 (Asymptotic behavior)**.** *Assume*[12] *that* $\lim_{p\to+\infty} H_\Delta(p) = 0$ *and* $\sigma > 0$. *The asymptotic behavior for* θ_Δ *is:*

$$\lim_{T\to+\infty} \theta_\Delta(0,q) = \Delta \sum_{q'\in\{\Delta,2\Delta,\dots,q\}} H_\Delta^{-1}\left(\frac{1}{2}\gamma^2\sigma^2 q'^2\right) = \theta_\Delta^\infty(q)$$

The resulting asymptotic behavior for the optimal quote is:

$$\lim_{T\to+\infty} (\delta^*_\Delta)_{t=0} = \delta^*_\Delta\left(H_\Delta^{-1}\left(\frac{1}{2}\gamma^2\sigma^2 q_0^2\right)\right) = \delta^{*\infty}_\Delta$$

Proof:

Let us define[13] for $q \in \{0, \Delta, 2\Delta, \dots, q_0\}$:

$$\theta_\Delta^\infty(q) = \Delta \sum_{q'\in\{\Delta,2\Delta,\dots,q\}} H_\Delta^{-1}\left(\frac{1}{2}\gamma^2\sigma^2 q'^2\right)$$

Let us define for $t \in [0,T]$ and $q \in \{0, \Delta, \dots, q_0\}$, $\theta^r_\Delta(t,q) = \theta_\Delta(T-t,q)$. Then, we want to prove that:

$$\forall q \in \{0, \Delta, \dots, q_0\}, \quad \lim_{t\to+\infty} \theta^r_\Delta(t,q) = \theta_\Delta^\infty(q)$$

We proceed by induction. The result is true for $q = 0$. Let us suppose that the result is true for $q - \Delta$ for some $q \in \{\Delta, \dots, q_0\}$.
Then:

$$\forall \epsilon > 0, \exists t_{q-\Delta}, \forall t \geq t_{q-\Delta}, |\theta^r_\Delta(t, q-\Delta) - \theta_\Delta^\infty(q-\Delta)| \leq \epsilon$$

Now, since H_Δ is a strictly decreasing function we have for $t \geq t_{q-\Delta}$:

$$-\frac{1}{2}\gamma^2\sigma^2 q^2 + H_\Delta\left(\frac{\theta^r_\Delta(t,q) - \theta_\Delta^\infty(q-\Delta) + \epsilon}{\Delta}\right)$$
$$\leq \gamma\partial_t\theta^r_\Delta(t,q) \leq -\frac{1}{2}\gamma^2\sigma^2 q^2 + H_\Delta\left(\frac{\theta^r_\Delta(t,q) - \theta_\Delta^\infty(q-\Delta) - \epsilon}{\Delta}\right)$$

[11]One may wonder why we choose a risk aversion parameter $\gamma = 0.001$. This figure seems small but it has in fact an important impact since the shares are sold by groups of 50.

[12]This is guaranteed if $\lim_{\delta\to+\infty} \delta\Lambda_\Delta(\delta) = 0$.

[13]This is well-defined because of the assumptions on H_Δ and σ, and because $\gamma > 0$. In the risk-neutral case, there is indeed no upper bound for the optimal quotes when $T \to \infty$. This constitutes an important difference between our model and the model of Bayraktar and Ludkovski [13].

or equivalently:

$$H_\Delta\left(\frac{\theta^r_\Delta(t,q)-\theta^\infty_\Delta(q-\Delta)+\epsilon}{\Delta}\right)-H_\Delta\left(\frac{\theta^\infty_\Delta(q)-\theta^\infty_\Delta(q-\Delta)}{\Delta}\right)$$

$$\leq \gamma\partial_t\theta^r_\Delta(t,q)\leq H_\Delta\left(\frac{\theta^r_\Delta(t,q)-\theta^\infty_\Delta(q-\Delta)-\epsilon}{\Delta}\right)-H_\Delta\left(\frac{\theta^\infty_\Delta(q)-\theta^\infty_\Delta(q-\Delta)}{\Delta}\right)$$

Hence, for $t\geq t_{q-\Delta}$:

$$\theta^r_\Delta(t,q)>\theta^\infty_\Delta(q)+\epsilon\Rightarrow\partial_t\theta^r_\Delta(t,q)<0$$

and

$$\theta^r_\Delta(t,q)<\theta^\infty_\Delta(q)-\epsilon\Rightarrow\partial_t\theta^r_\Delta(t,q)>0$$

As a consequence, if there exists $t_q\geq t_{q-\Delta}$ such that $|\theta^r_\Delta(t_q,q)-\theta^\infty_\Delta(q)|\leq\epsilon$ then, $\forall t\geq t_q, |\theta^r_\Delta(t,q)-\theta^\infty_\Delta(q)|\leq\epsilon$.

In particular, if $|\theta^r_\Delta(t_{q-\Delta},q)-\theta^\infty_\Delta(q)|\leq\epsilon$ then, $\forall t\geq t_{q-\Delta}, |\theta^r_\Delta(t,q)-\theta^\infty_\Delta(q)|\leq\epsilon$.

Now, if $\theta^r_\Delta(t_{q-\Delta},q)>\theta^\infty_\Delta(q)+\epsilon$, then there are two possibilities. The first one is that the function $t\geq t_{q-\Delta}\mapsto\theta^r_\Delta(t,q)$ is decreasing and in that case it is bounded from below by $\theta^\infty_\Delta(q)-\epsilon$ and must converge. Since $\lim_{t\to+\infty}\theta^r_\Delta(t,q-\Delta)=\theta^\infty_\Delta(q-\Delta)$, the only possible limit for $\theta^r_\Delta(t,q)$ is $\theta^\infty_\Delta(q)$. The second possibility is that $t\geq t_{q-\Delta}\mapsto\theta^r_\Delta(t,q)$ is not a decreasing function and in that case there must exists $t_q\geq t_{q-\Delta}$ such that $\theta^r_\Delta(t_q,q)\leq\theta^\infty_\Delta(q)+\epsilon$. Since $\theta^r_\Delta(t_q,q)\geq\theta^\infty_\Delta(q)-\epsilon$, we now that $\forall t\geq t_q, |\theta^r_\Delta(t,q)-\theta^\infty_\Delta(q)|\leq\epsilon$.

Finally, if $\theta^r_\Delta(t_{q-\Delta},q)<\theta^\infty_\Delta(q)-\epsilon$, then there are two possibilities. The first one is that the function $t\geq t_{q-\Delta}\mapsto\theta^r_\Delta(t,q)$ is increasing and in that case it is bounded from above by $\theta^\infty_\Delta(q)+\epsilon$ and must converge. Since $\lim_{t\to+\infty}\theta^r_\Delta(t,q-\Delta)=\theta^\infty_\Delta(q-\Delta)$, the only possible limit for $\theta^r_\Delta(t,q)$ is $\theta^\infty_\Delta(q)$. The second possibility is that $t\geq t_{q-\Delta}\mapsto\theta^r_\Delta(t,q)$ is not an increasing function and in that case there must exists $t_q\geq t_{q-\Delta}$ such that $\theta^r_\Delta(t_q,q)\geq\theta^\infty_\Delta(q)-\epsilon$. Since $\theta^r_\Delta(t_q,q)\leq\theta^\infty_\Delta(q)+\epsilon$, we now that $\forall t\geq t_q, |\theta^r_\Delta(t,q)-\theta^\infty_\Delta(q)|\leq\epsilon$.

The conclusion is that $\limsup_{t\to+\infty}|\theta^r_\Delta(t,q)-\theta^\infty_\Delta(q)|\leq\epsilon$. Sending ϵ to 0, we get the result for θ_Δ.

The result for the optimal quote is then straightforward. □

These asymptotic formulae deserve some comments. Firstly, regarding the above discussion on monotonicity, we know that H_Δ is a decreasing function and we then see on the asymptotic limit that $\theta^\infty_\Delta(\cdot)$ is either a decreasing function (when $\frac{1}{2}\gamma^2\sigma^2\Delta^2>H_\Delta(0)$) or a function that is first increasing and then decreasing (otherwise). Secondly, coming to the optimal quotes and the role of the parameters, we can analyze the way $\delta^{*\infty}_\Delta$ depends on σ, γ and Λ_Δ. The best way to proceed is to use the expression for H_Δ found in Lemma 1 and to notice that an equivalent way to define $\delta^{*\infty}_\Delta$ is through the following implicit characterization:

$$\frac{1}{2}\gamma\sigma^2q_0^2=\Delta\frac{\Lambda_\Delta(\delta^{*\infty}_\Delta)^2}{\gamma\Delta\Lambda_\Delta(\delta^{*\infty}_\Delta)-\Lambda'_\Delta(\delta^{*\infty}_\Delta)}$$

It is then straightforward to see that $\delta^{*\infty}_\Delta$ decreases as σ increases. An increase in σ corresponds to an increase in price risk and this provides the trader with an incentive to speed up the execution process. Therefore, it is natural that the asymptotic quote is a decreasing

function of σ.
Differentiating the above expression with respect to γ, we see that the asymptotic quote decreases as the risk aversion increases. An increase in risk aversion forces indeed the trader to reduce both non-execution risk and price risk and this leads to posting orders with lower prices.
Now, if one replaces the intensity function Λ_Δ by $\lambda\Lambda_\Delta$ where $\lambda > 1$, then it results in an increase in $\delta_\Delta^{*\infty}$. This is natural because when the rate of arrival of market orders increases, the trader is more likely to liquidate his shares faster and posting deeper into the book allows for larger profits.

In addition to the asymptotic regime, we can consider different sizes Δ of orders. We provide below a numerical approximation of the quotes for two different sizes, when the intensity function is scaled by the size of the orders:

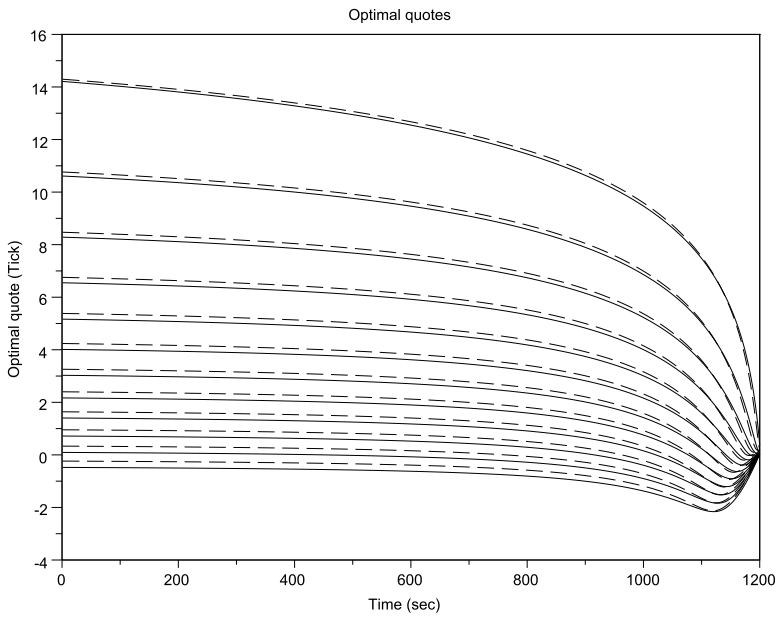

Figure 3: Optimal quotes for $q \in \{50, 100, \ldots, 600\}$, for $q_0 = 600$, $T = 1200$ (s), $\sigma = 0.3$ (Tick.s$^{-\frac{1}{2}}$), $A = 0.1$ (s^{-1}), $k = 0.3$ (Tick^{-1}), $\gamma = 0.001$ (Tick^{-1}) and $\ell = 3$ (Tick). Line: $\Lambda(\delta) = Ae^{-k\delta}$ and $\Delta = 50$. Dotted line: $\Lambda(\delta) = 2Ae^{-k\delta}$ and $\Delta = 25$.

We see on Figure 3, that there is little difference between the two cases we considered. This is linked to the existence of a limit regime as $\Delta \to 0$ and the next section is dedicated to its analysis. The limiting equation for θ_Δ will turn out to be a classical equation in optimal liquidation theory (see section 5).

4 Limit regime $\Delta \to 0$

In the preceding section the size of the order sent by the trader was fixed at Δ, a size that is supposed to be small with respect to q_0. As a consequence, the question of the limiting behavior when Δ tends to 0 is relevant[14]. To that purpose we need to make an assumption

[14]Although this is not recalled, it is assumed that Δ is always chosen as a fraction of q_0.

on the behavior of the intensity function with respect to the order size Δ. The "right" scaling (already used above for the numerics underlying Figure 3) is to suppose that

$$\Lambda_\Delta(\delta) = \frac{\Lambda(\delta)}{\Delta}$$

With this scaling assumption, along with additional technical hypotheses, our goal is to prove the following Theorem that is rather technical and echoes the results obtained by [13], here with risk aversion (*i.e.* $\gamma > 0$) whereas [13] deals with a risk-neutral case:

Theorem 2 (Limit regime $\Delta \to 0$)**.** *Let us suppose that:*

- $\Lambda(\delta)\Lambda''(\delta) < 2\Lambda'(\delta)^2$
- $\lim_{\delta\to+\infty} \delta\Lambda(\delta) = 0$
- *ℓ is a continuous function*

For a given $\Delta > 0$, let us define θ^c_Δ on $[0,T] \times [0,q_0]$ by:

$$\theta^c_\Delta(t,q) = \begin{cases} \theta_\Delta(t,0), & \text{if } q = 0 \\ \theta_\Delta\left(t,(k+1)\Delta\right), & \text{if } q \in (k\Delta,(k+1)\Delta] \end{cases}$$

Then θ^c_Δ converges uniformly toward a continuous function $\theta : [0,T] \times [0,q_0] \to \mathbb{R}$ that solves in the viscosity sense:

$$\begin{cases} -\gamma\partial_t\theta(t,q) + \frac{1}{2}\gamma^2\sigma^2q^2 - H\left(\partial_q\theta(t,q)\right) = 0, & \text{on } [0,T) \times (0,1] \\ \theta(t,q) = 0, & \text{on } [0,T] \times \{0\} \\ \theta(t,q) = -\ell(q)q, & \text{on } \{T\} \times [0,q_0] \end{cases}$$

where $H(p) = \gamma \sup_\delta \Lambda(\delta)(\delta - p)$ and where the terminal condition and the boundary condition are in fact satisfied is the classical sense.

To prove this theorem, we first need to study H and the convergence of the Hamiltonian functions H_Δ towards H. We start with a counterpart of Lemma 1 that requires $\Lambda(\delta)\Lambda''(\delta) < 2\Lambda'(\delta)^2$.

Lemma 2. *Let us define $L(p,\delta) = \Lambda(\delta)\left(\delta - p\right)$.*

$\forall p \in \mathbb{R}$, $\delta \mapsto L(p,\delta)$ reaches its maximum at $\tilde{\delta}^*(p)$ uniquely characterized by:

$$\tilde{\delta}^*(p) + \frac{\Lambda(\tilde{\delta}^*(p))}{\Lambda'(\tilde{\delta}^*(p))} = p$$

Moreover, $p \mapsto \tilde{\delta}^*(p)$ is a C^1 function.

Subsequently H is a C^1 function with:

$$H_\Delta(p) = \gamma\frac{\Lambda(\tilde{\delta}^*(p))^2}{-\Lambda'(\tilde{\delta}^*(p))}$$

Proof: The proof is similar to the proof of Lemma 1.

Now we can state a result about convergence that also provides a uniform bound for the hamiltonian functions:

Lemma 3. *H_Δ converges locally uniformly towards H when $\Delta \to 0$ with $\forall p \in \mathbb{R}, H_\Delta(p) \leq H(p)$.*

Proof:

For a fixed $x \geq 0$, the function $f : \Delta \in \mathbb{R}_+ \mapsto \frac{1-e^{-\Delta x}}{\Delta}$ is decreasing function with $f(0) = x$.

Hence,

$$\forall 0 < \Delta' < \Delta, \quad \sup_{\delta \geq p} \Lambda(\delta) \frac{1 - e^{-\gamma\Delta(\delta - p)}}{\Delta} \leq \sup_{\delta \geq p} \Lambda(\delta) \frac{1 - e^{-\gamma\Delta'(\delta - p)}}{\Delta'} \leq \gamma \sup_{\delta \geq p} \Lambda(\delta)(\delta - p)$$

This gives:

$$H_\Delta(p) \leq H_{\Delta'}(p) \leq H(p)$$

Now, because H is continuous, using Dini's theorem, if we prove that convergence is pointwise, convergence will be locally uniform. We are then left with the need to prove pointwise convergence of H_Δ toward H. Using Lemma 1 and Lemma 2 we see that it is sufficient to prove that $\tilde{\delta}^*_\Delta$ converges pointwise towards $\tilde{\delta}^*$.

But the sequence of functions $f_\Delta(x) = x - \frac{1}{\gamma\Delta} \log\left(1 - \gamma\Delta \frac{\Lambda(x)}{\Lambda'(x)}\right)$ is an increasing sequence of increasing functions and hence by the unique characterizations of $\tilde{\delta}^*_\Delta(p)$ and $\tilde{\delta}^*(p)$, we see that $\tilde{\delta}^*_\Delta(p)$ increases as Δ decreases to 0 and is bounded from above by $\tilde{\delta}^*(p)$ which is the only possible limit. Hence $\tilde{\delta}^*_\Delta(p) \to \tilde{\delta}^*(p)$ as $\Delta \to 0$ and this proves the result. □

Now, we will provide a uniform bound for the θ_Δ that will be important in the proof of Theorem 2.

Proposition 5 (Bounds for θ_Δ)**.** $\forall t \in [0, T], \forall q \in \{0, \Delta, \ldots, q_0\}$,

$$-\ell(q_0)q_0 - \frac{1}{2}\gamma\sigma^2 q_0^2 (T - t) \leq \theta_\Delta(t, q) \leq \frac{1}{\gamma} H(0)(T - t)$$

Proof:

To prove these inequalities, we use the comparison principle enounced above.

If $\overline{\theta}_\Delta(t, q) = \frac{1}{\gamma} H_\Delta(0)(T - t)$ then:

$$\overline{\theta}_\Delta(T, q) = 0 \geq -\ell(q)q \qquad \overline{\theta}_\Delta(t, 0) = \frac{1}{\gamma} H_\Delta(0)(T - t) \geq 0$$

and

$$\gamma \partial_t \overline{\theta}_\Delta(t, q) - \frac{1}{2}\gamma^2\sigma^2 q^2 + H_\Delta\left(\frac{\overline{\theta}_\Delta(t, q) - \overline{\theta}_\Delta(t, q - \Delta)}{\Delta}\right)$$
$$= -\gamma \frac{1}{\gamma} H_\Delta(0) - \frac{1}{2}\gamma^2\sigma^2 q^2 + H_\Delta(0) = -\frac{1}{2}\gamma^2\sigma^2 q^2 \leq 0$$

Hence $\theta_\Delta(t, q) \leq \overline{\theta}_\Delta(t, q) = \frac{1}{\gamma} H_\Delta(0)(T - t)$.

The uniform upper bound is then obtained using Lemma 3.

Now, if $\underline{\theta}_\Delta(t, q) = -\ell(q_0)q_0 - \frac{1}{2}\gamma\sigma^2 q_0^2 (T - t)$ then:

$$\underline{\theta}_\Delta(T, q) = -\ell(q_0)q_0 \leq -\ell(q)q \qquad \underline{\theta}_\Delta(t, 0) = \frac{1}{\gamma} - \ell(q_0)q_0 - \frac{1}{2}\gamma^2\sigma^2 q_0^2 (T - t) \leq 0$$

and

$$\gamma\partial_t\underline{\theta}_\Delta(t,q) - \frac{1}{2}\gamma^2\sigma^2q^2 + H_\Delta\left(\frac{\underline{\theta}_\Delta(t,q) - \underline{\theta}_\Delta(t,q-\Delta)}{\Delta}\right)$$
$$= \frac{1}{2}\gamma^2\sigma^2q_0^2 - \frac{1}{2}\gamma^2\sigma^2q^2 + H_\Delta(0) \geq \frac{1}{2}\gamma^2\sigma^2(q_0^2 - q^2) \geq 0$$

Hence $\theta_\Delta(t,q) \geq \underline{\theta}_\Delta(t,q) = -\ell(q_0)q_0 - \frac{1}{2}\gamma\sigma^2q_0^2(T-t)$. □

We are now ready to start the proof of Theorem 2.

Proof of Theorem 2:

We first introduce the following half-relaxed limit functions:

$$\overline{\theta}(t,x) = \limsup_{j\to+\infty}\limsup_{\Delta\to 0}\sup\{\theta^c_\Delta(t',q'), \quad |t'-t| + |q'-q| \leq \frac{1}{j}\}$$

$$\underline{\theta}(t,x) = \liminf_{j\to+\infty}\liminf_{\Delta\to 0}\inf\{\theta^c_\Delta(t',q'), \quad |t'-t| + |q'-q| \leq \frac{1}{j}\}$$

$\overline{\theta}$ and $\underline{\theta}$ are respectively upper semi-continuous and lower semi-continuous and the goal of the proof is to show that they are equal to one another and solution of the above partial differential equation.

Step 1: $\overline{\theta}$ and $\underline{\theta}$ are respectively subsolution and supersolution of

$$\begin{cases} -\gamma\partial_t\theta(t,q) + \frac{1}{2}\gamma^2\sigma^2q^2 - H\left(\partial_q\theta(t,q)\right) = 0, & \text{on } [0,T)\times(0,1] \\ \theta(t,q) = 0, & \text{on } [0,T]\times\{0\} \\ \theta(t,q) = -\ell(q)q, & \text{on } \{T\}\times[0,q_0] \end{cases}$$

where the conditions are to be understood in the viscosity sense.

To prove this point, let us consider a test function $\phi \in C^1([0,T]\times[0,q_0])$ and (t^*,q^*) a local maximum of $\overline{\theta} - \phi$. Without loss of generality, we can assume that $\phi(t^*,q^*) = \overline{\theta}(t^*,q^*)$ and consider $r > 0$ such that:

- the maximum is global on the ball of radius r centered in (t^*,q^*).
- outside of this ball $\phi \geq 2\sup_\Delta \|\theta_\Delta\|_\infty$ – this value being finite because of the uniform bound obtained in the above Proposition.

Following Barles-Souganidis methodology [12], we know that there exists a sequence $(\Delta_n, t_n, q_n)_n$ such that:

- $\Delta_n \to 0$, $(t_n,q_n) \to (t^*,q^*)$
- $\theta^c_{\Delta_n}(t_n,q_n) \to \overline{\theta}(t^*,q^*)$
- (t_n,q_n) is a global maximum of $\theta^c_{\Delta_n} - \phi$

Now, because of the definition of $\theta^c_{\Delta_n}$, if $(t^*,q^*) \in [0,T)\times(0,q_0]$ we can always suppose that $q_n \geq \Delta_n$ and $t_n \neq T$ and then, by definition of θ_{Δ_n}:

$$-\gamma\partial_t\theta_{\Delta_n}(t_n,q_n) + \frac{1}{2}\gamma^2\sigma^2q_n^2 - H_{\Delta_n}\left(\frac{\theta_{\Delta_n}(t_n,q_n) - \theta_{\Delta_n}(t_n,q_n-\Delta_n)}{\Delta_n}\right) = 0$$

Because H_{Δ_n} is decreasing we have, by definition of (t_n,q_n):

$$-H_{\Delta_n}\left(\frac{\theta_{\Delta_n}(t_n,q_n)-\theta_{\Delta_n}(t_n,q_n-\Delta_n)}{\Delta_n}\right)\geq -H_{\Delta_n}\left(\frac{\phi(t_n,q_n)-\phi(t_n,q_n-\Delta_n)}{\Delta_n}\right)$$

Similarly, since $t_n < T$ we have, for h sufficiently small:

$$\theta_{\Delta_n}(t_n+h,q_n)-\theta_{\Delta_n}(t_n,q_n)\leq \phi(t_n+h,q_n)-\phi(t_n,q_n)$$

Hence:

$$\partial_t\theta_{\Delta_n}(t_n,q_n)\leq \partial_t\phi(t_n,q_n)$$

These inequalities give:

$$-\gamma\partial_t\phi(t_n,q_n)+\frac{1}{2}\gamma^2\sigma^2q_n^2-H_{\Delta_n}\left(\frac{\phi(t_n,q_n)-\phi(t_n,q_n-\Delta_n)}{\Delta_n}\right)\leq 0$$

Using now the convergence of (t_n,q_n) towards (t^*,q^*) and the local uniform convergence of H_{Δ_n} towards H we eventually obtain the desired inequality:

$$-\gamma\partial_t\phi(t^*,q^*)+\frac{1}{2}\gamma^2\sigma^2{q^*}^2-H\left(\partial_q\phi(t^*,q^*)\right)\leq 0$$

We see that the boundaries corresponding to $q=q_0$ and $t=0$ play no role. However, we need to consider the cases $t^*=T$ and $q^*=0$.

If $t^*=T$ and $q^*\neq 0$ then there are two cases. If there are infinitely many indices n such that $t_n<T$ then the preceding proof still works. Otherwise, for all n sufficiently large, $\theta_{\Delta_n}(t_n,q_n)=-\ell(q_n)q_n$ and hence, passing to the limit, $\overline{\theta}(t^*,q^*)=-\ell(q^*)q^*$.
Eventually, we indeed have that:

$$\min(-\gamma\partial_t\phi(T,q^*)+\frac{1}{2}\gamma^2\sigma^2{q^*}^2-H\left(\partial_q\phi(T,q^*)\right),\overline{\theta}(T,q^*)+\ell(q^*)q^*)\leq 0$$

If $q^*=0$ then there are also two cases. If there are infinitely many indices n such that $q_n^*>0$ and $t_n<T$ then the initial proof still works. Otherwise, for n sufficiently large $\theta_{\Delta_n}(t_n,q_n)=-\ell(q_n)q_n$ or $\theta_{\Delta_n}(t_n,q_n)=0$ and hence, passing to the limit, we obtain $\overline{\theta}(t^*,q^*)=0$.

Eventually, we indeed have that:

$$\min(-\gamma\partial_t\phi(t^*,0)-H\left(\partial_q\phi(t^*,0)\right),\overline{\theta}(t^*,0))\leq 0$$

We have proved that $\overline{\theta}$ is a subsolution of the equation in the viscosity and one can similarly prove that $\underline{\theta}$ is a supersolution.

Step 2: $\forall q\in[0,q_0],\quad \overline{\theta}(T,q)=\underline{\theta}(T,q)=-\ell(q)q.$

We consider the test function $\phi(t,q)=C_\epsilon(T-t)+\frac{1}{\epsilon}(q-q_{ref})^2$ where $q_{ref}\in[0,q_0]$ is fixed, where $\epsilon>0$ is a constant and where C_ϵ is a constant that depends on ϵ and will be fixed later.

Let (t_ϵ,q_ϵ) be a maximum point of $\overline{\theta}-\phi$ on $[0,T]\times[0,q_0]$. Then:

$$\overline{\theta}(T,q_{ref})\leq\overline{\theta}(t_\epsilon,q_\epsilon)-C_\epsilon(T-t_\epsilon)-\frac{1}{\epsilon}(q_\epsilon-q_{ref})^2$$

This inequality gives $q_\epsilon\to q_{ref}$ as $\epsilon\to 0$.

Now,

$$-\gamma\partial_t\phi(t_\epsilon,q_\epsilon)+\frac{1}{2}\gamma^2\sigma^2{q_\epsilon}^2-H\left(\partial_q\phi(t_\epsilon,q_\epsilon)\right)=\gamma C_\epsilon+\frac{1}{2}\gamma^2\sigma^2{q_\epsilon}^2-H\left(\frac{2}{\epsilon}(q_\epsilon-q_{ref})\right)$$

$$\geq\gamma C_\epsilon-H\left(-\frac{2q_0}{\epsilon}\right)$$

Hence, if we consider $C_\epsilon=\frac{1}{\gamma}H\left(-\frac{2q_0}{\epsilon}\right)+1$, we see that we must have either $t_\epsilon=T$ and $\overline{\theta}(t_\epsilon,q_\epsilon)\leq-\ell(q_\epsilon)q_\epsilon$ or (for sufficiently small ϵ, only in the case where $q_{ref}=0$), $\overline{\theta}(t_\epsilon,q_\epsilon)\leq 0$.

If $q_{ref}\neq 0$, we then write, for ϵ sufficiently small:

$$\overline{\theta}(T,q_{ref})\leq\overline{\theta}(t_\epsilon,q_\epsilon)-C_\epsilon(T-t_\epsilon)-\frac{1}{\epsilon}(q_\epsilon-q_{ref})^2\leq\overline{\theta}(t_\epsilon,q_\epsilon)\leq-\ell(q_\epsilon)q_\epsilon$$

Sending ϵ to 0 we obtain

$$\overline{\theta}(T,q_{ref})\leq-\ell(q_{ref})q_{ref}$$

If $q_{ref}=0$, then we know from the above two inequalities that:

$$\overline{\theta}(T,q_{ref})\leq\overline{\theta}(t_\epsilon,q_\epsilon)-C_\epsilon(T-t_\epsilon)-\frac{1}{\epsilon}(q_\epsilon-q_{ref})^2\leq\overline{\theta}(t_\epsilon,q_\epsilon)\leq\max(-\ell(q_\epsilon)q_\epsilon,0)=0$$

Hence $\forall q\in[0,q_0],\overline{\theta}(T,q)\leq\ell(q)q$ and the same proof works for the supersolution[15] to get $\forall q\in[0,q_0],\underline{\theta}(T,q)\geq\ell(q)q$.

As a consequence $\forall q\in[0,q_0],\overline{\theta}(T,q)\leq-\ell(q)q\leq\underline{\theta}(T,q)$ and eventually, because $\underline{\theta}(T,q)\leq\overline{\theta}(T,q)$: $\forall q\in[0,q_0],\overline{\theta}(T,q)=-\ell(q)q=\underline{\theta}(T,q)$.

Step 3: $\forall t\in[0,T],\quad\underline{\theta}(t,0)=0$

Concerning the boundary condition corresponding to $q=0$ we can apply the same ideas but only to the supersolution $\underline{\theta}$.

Let us consider indeed $t_{ref}\in[0,T)$ and the test function $\phi(t,q)=-C_\epsilon q-\frac{1}{\epsilon}(t-t_{ref})^2$. Then, let (t_ϵ,q_ϵ) be a minimum point of $\underline{\theta}-\phi$ on $[0,T]\times[0,q_0]$. We have:

$$\underline{\theta}(t_{ref},0)\geq\underline{\theta}(t_\epsilon,q_\epsilon)+C_\epsilon q_\epsilon+\frac{1}{\epsilon}(t_\epsilon-t_{ref})^2$$

This inequality gives $t_\epsilon\to t_{ref}$ as $\epsilon\to 0$.
Now,

$$-\gamma\partial_t\phi(t_\epsilon,q_\epsilon)+\frac{1}{2}\gamma^2\sigma^2{q_\epsilon}^2-H\left(\partial_q\phi(t_\epsilon,q_\epsilon)\right)=2\gamma\frac{t-t_{ref}}{\epsilon}+\frac{1}{2}\gamma^2\sigma^2{q_\epsilon}^2-H\left(-C_\epsilon\right)$$

$$\leq 2\gamma\frac{T}{\epsilon}+\frac{1}{2}\gamma^2\sigma^2{q_0}^2-H\left(-C_\epsilon\right)$$

Since $\lim_{p\to-\infty}H(p)=+\infty$, we can always choose C_ϵ so that the above expression is strictly negative.

[15]The only difference is that we have to pass to the limit in the case $q_{ref}=0$ to obtain the conclusion.

As a consequence, for ϵ sufficiently small, we must have $q_\epsilon = 0$ and $\underline{\theta}(t_\epsilon, 0) \geq 0$.

Consequently:

$$\underline{\theta}(t_{ref}, 0) \geq \underline{\theta}(t_\epsilon, q_\epsilon) + C_\epsilon q_\epsilon + \frac{1}{\epsilon}(t_\epsilon - t_{ref})^2 \geq 0$$

This result being already true for $t_{ref} = T$, we have that $\underline{\theta}(t, 0) \geq 0, \forall t \in [0, T]$ and in fact $\underline{\theta}(t, 0) = 0, \forall t \in [0, T]$ because of the definition of $\underline{\theta}(t, 0)$.

Step 4: $\forall t \in [0, T]$, there exists a sequence (t_n, q_n) such that $t_n \neq t$, $q_n \neq 0$, $(t_n, q_n) \to (t, 0)$ and $\underline{\theta}(t_n, q_n) \to 0$

To prove this claim, we prove that $g_\Delta(t) = \theta_\Delta(T-t, \Delta)$ converges uniformly (in t) toward 0 as $\Delta \to 0$.

By definition $g_\Delta(0) = -\ell(\Delta)\Delta$ and $g'_\Delta(t) = -\frac{1}{2}\gamma\sigma^2\Delta^2 + \frac{1}{\gamma}H_\Delta\left(\frac{g_\Delta(t)}{\Delta}\right)$.

The stationary state of the above ODE is $g^\infty_\Delta = \Delta H^{-1}_\Delta(\frac{1}{2}\gamma^2\sigma^2\Delta^2)$ and $\Delta H^{-1}_\Delta(\frac{1}{2}\gamma^2\sigma^2\Delta^2) \geq \Delta H^{-1}_{\Delta'}(\frac{1}{2}\gamma^2\sigma^2\Delta^2)$ as soon as $\Delta < \Delta'$. Hence g^∞_Δ is positive for Δ sufficiently small. As a consequence, because H_Δ is decreasing, for Δ sufficiently small we know that g_Δ is increasing, bounded from below by $-\ell(\Delta)\Delta$ and bounded from above by g^∞_Δ.

Now $g'_\Delta(t) \leq \frac{1}{\gamma}H\left(\frac{g_\Delta(t)}{\Delta}\right)$. Hence,

$$\int_0^{g_\Delta(t)} \frac{\gamma}{H\left(\frac{y}{\Delta}\right)}dy \leq \int_{-\ell(\Delta)\Delta}^{g_\Delta(t)} \frac{\gamma}{H\left(\frac{y}{\Delta}\right)}dy \leq t$$

and this gives $g_\Delta(t) \leq \Delta G^{-1}\left(\frac{T}{\gamma\Delta}\right)$, where $G(x) = \int_0^x \frac{1}{H(y)}dy$.

Consequently,

$$-\ell(\Delta)\Delta \leq g_\Delta(t) \leq \Delta G^{-1}\left(\frac{T}{\gamma\Delta}\right)$$

and it is sufficient to have $\lim_{x\to+\infty} \frac{G(x)}{x} = +\infty$.
But this is true as soon as $\lim_{x\to+\infty} H(x) = 0$, which is guaranteed by $\lim_{\delta\to+\infty} \delta\Lambda(\delta) = 0$.

Step 5: Comparison principle: $\overline{\theta} \leq \underline{\theta}$.

Let us consider $\alpha > 0$ and the maximum $M = \max_{(t,q)\in[0,T]\times[0,q_0]} \overline{\theta}(t, q) - \underline{\theta}(t, q) - \alpha(T-t)$. If $m = \max_{t\in[0,T]} \overline{\theta}(t, 0) - \underline{\theta}(t, 0) - \alpha(T - t) < M$ then we distinguish two cases.

Case 1: $m \leq 0$.

We introduce $\Phi_\epsilon(t, q, t', q') = \overline{\theta}(t, q) - \underline{\theta}(t', q') - \alpha(T - t) - \frac{(q-q')^2}{\epsilon} - \frac{(t-t')^2}{\epsilon}$.

Let us consider $(t_\epsilon, q_\epsilon, t'_\epsilon, q'_\epsilon)$ a maximum point of Φ_ϵ. We have $M \leq \Phi_\epsilon(t_\epsilon, q_\epsilon, t'_\epsilon, q'_\epsilon)$ and we are going to prove that $\liminf_{\epsilon\to 0} \Phi_\epsilon(t_\epsilon, q_\epsilon, t'_\epsilon, q'_\epsilon) \leq 0$.

We have $\Phi_\epsilon(t_\epsilon, q_\epsilon, t_\epsilon, q_\epsilon) \leq \Phi_\epsilon(t_\epsilon, q_\epsilon, t'_\epsilon, q'_\epsilon)$ and hence $\frac{(q_\epsilon-q'_\epsilon)^2}{\epsilon} + \frac{(t_\epsilon-t'_\epsilon)^2}{\epsilon}$ is bounded. As a consequence, $t_\epsilon - t'_\epsilon \to 0$ and $q_\epsilon - q'_\epsilon \to 0$.

Now, if for all ϵ sufficiently small we have $(t_\epsilon, q_\epsilon, t'_\epsilon, q'_\epsilon) \in [0,T) \times (0,q_0] \times [0,T) \times (0,q_0]$ then we have:

$$-2\gamma\frac{t_\epsilon - t'_\epsilon}{\epsilon} + \gamma\alpha + \frac{1}{2}\gamma^2\sigma^2 q_\epsilon^2 - H(2\frac{q_\epsilon - q'_\epsilon}{\epsilon}) \leq 0$$

and

$$-2\gamma\frac{t_\epsilon - t'_\epsilon}{\epsilon} + \frac{1}{2}\gamma^2\sigma^2 {q'_\epsilon}^2 - H(2\frac{q_\epsilon - q'_\epsilon}{\epsilon}) \geq 0$$

Hence $\gamma\alpha + \frac{1}{2}\gamma^2\sigma^2(q_\epsilon^2 - {q'_\epsilon}^2) \leq 0$ and this is a contradiction as we send ϵ to 0.

The consequence is that there exists a sequence $\epsilon_n \to 0$ such that $(t_{\epsilon_n}, q_{\epsilon_n}, t'_{\epsilon_n}, q'_{\epsilon_n})$ verifies $t_{\epsilon_n} = T, \forall n$ or $q_{\epsilon_n} = 0, \forall n$ or $t'_{\epsilon_n}, \forall n$ or $q'_{\epsilon_n}, \forall n$.

Then

$$\limsup_{n\to+\infty} \Phi_{\epsilon_n}(t_{\epsilon_n}, q_{\epsilon_n}, t'_{\epsilon_n}, q'_{\epsilon_n}) \leq \limsup_{n\to+\infty} \overline{\theta}(t_{\epsilon_n}, q_{\epsilon_n}) - \underline{\theta}(t'_{\epsilon_n}, q'_{\epsilon_n}) - \alpha(T - t_{\epsilon_n})$$

$$\leq \max_{(t,q)\in(\{T\}\times[0,q_0])\cup([0,T]\times\}q_0\})} \overline{\theta}(t,q) - \underline{\theta}(t,q) - \alpha(T-t) \leq \max(0,m) \leq 0$$

Hence in that case $M \leq 0$.

Case 2: $m > 0$.

In that case, we replace $\underline{\theta}$ by $\underline{\theta} + m$ in the above case and we obtain, instead of $M \leq 0$, the inequality $M \leq m$ which contradicts our hypothesis.

It remains to consider the case $m = M$. We know then that the maximum M is attained at a point $(t_{max}, 0)$ and we suppose that $M > 0$.

From Step 4, we consider a sequence (t_n, q_n) such that $t_n \neq t_{max}$, $q_n \neq 0$, $(t_n, q_n) \to (t_{max}, 0)$ and $\underline{\theta}(t_n, q_n) \to 0$.

We define:

$$\Psi_n(t,q,t',q') = \overline{\theta}(t,q) - \underline{\theta}(t',q') - \alpha(T-t) - \frac{(t-t')^2}{|t_n - t_{max}|} - \left(\frac{q'-q}{q_n} - 1\right)^2$$

This function attains its maximum at a point $(t_n^*, q_n^*, {t'}_n^*, {q'}_n^*)$. We first consider the inequality $\Psi_n(t_{max}, 0, t_n, q_{max}) \leq \Psi_n(t_n^*, q_n^*, {t'}_n^*, {q'}_n^*)$:

$$\overline{\theta}(t_{max}, 0) - \underline{\theta}(t_n, q_n) - \alpha(T - t_{max}) - |t_n - t_{max}|$$

$$\leq \overline{\theta}(t_n^*, q_n^*) - \underline{\theta}({t'}_n^*, {q'}_n^*) - \alpha(T - t_n^*) - \frac{(t_n^* - {t'}_n^*)^2}{|t_n - t_{max}|} - \left(\frac{{q'}_n^* - q_n^*}{q_n} - 1\right)^2$$

We then have $t_n^* - {t'}_n^* \to 0$ and ${q'}_n^* - q_n^* \to 0$.

Hence $\limsup_n \overline{\theta}(t_n^*, q_n^*) - \underline{\theta}({t'}_n^*, {q'}_n^*) - \alpha(T - t_n^*) \leq M$. Now, the maximum M is also given by $\lim_n \overline{\theta}(t_{max}, 0) - \underline{\theta}(t_n, q_n) - \alpha(T - t_{max})$ and we obtain that the penalization term $\left(\frac{{q'}_n^* - q_n^*}{q_n} - 1\right)^2$ converge to 0.

This gives ${q'}_n^* = q_n^* + q_n + o(q_n) > 0$.

Now, if we have infinitely many n such that ${t'}_n^* = T$ then:

$$M = \lim_n \overline{\theta}(t_{max}, 0) - \underline{\theta}(t_n, q_n) - \alpha(T - t_{max}) \leq \sup_{q \in [0, q_0]} \overline{\theta}(T, q) - \underline{\theta}(T, q) = 0$$

and this contradicts our hypothesis.

Otherwise, for all n sufficiently large:

$$-2\gamma \frac{t_n^* - {t'}_n^*}{|t_n - t_{max}|} + \frac{1}{2}\gamma^2\sigma^2 {q'}_n^{*\,2} - H\left(-\frac{2}{q_n}\frac{{q'}_n^* - q_n^*}{q_n} - 1\right) \geq 0$$

Now, going to the subsolution, if there are infinitely many n such that $t_n^* = T$, then:

$$M = \lim_n \overline{\theta}(t_{max}, 0) - \underline{\theta}(t_n, q_n) - \alpha(T - t_{max}) \leq \sup_{q \in [0, q_0]} \overline{\theta}(T, q) - \underline{\theta}(T, q) = 0$$

in contradiction with our hypothesis.

Else, if $q_n^* = 0$ and $\overline{\theta}(t_n^*, q_n^*) \leq 0$ for infinitely many n then we obtain $M = \lim_n \overline{\theta}(t_{max}, 0) - \underline{\theta}(t_n, q_n) - \alpha(T - t_{max}) - |t_n - t_{max}| \leq 0$ straightforwardly and this contradicts our hypothesis. Hence, the viscosity inequality must be satisfied and we get:

$$-2\gamma \frac{t_n^* - {t'}_n^*}{|t_n - t_{max}|} + \gamma\alpha + \frac{1}{2}\gamma^2\sigma^2 q_n^{*\,2} - H(-\frac{2}{q_n}\frac{{q'}_n^* - q_n^*}{q_n} - 1) \leq 0$$

Combining the two inequalities eventually leads to $\alpha\gamma \leq 0$ as $n \to \infty$ and this is a contradiction.

We have obtained that $M \leq 0$ and hence $\overline{\theta} - \underline{\theta} \leq \alpha T$ and, sending α to 0 we get $\overline{\theta} \leq \underline{\theta}$.

This proves that $\overline{\theta} = \underline{\theta}$ is in fact a continuous function that we call θ, solution of the PDE introduced above.

We have:

$$\theta(t, q) = \underline{\theta}(t, q) \leq \liminf_{\Delta \to 0} \theta_\Delta^c(t, q) \leq \limsup_{\Delta \to 0} \theta_\Delta^c(t, q) \leq \overline{\theta}(t, q) = \theta(t, q)$$

Hence $\theta(t, q) = \lim_{\Delta \to 0} \theta_\Delta^c(t, q)$ and, by the same token, $\lim_{\Delta \to 0, (t', q') \to (t, q)} \theta_\Delta^c(t', q') = \theta(t, q)$ so that the convergence is locally uniform and then uniform on the compact set $[0, T] \times [0, q_0]$. □

5 Link with Almgren-Chriss and interpretation of Λ for negative δ

5.1 Back to Almgren-Chriss

In the above section we proved that θ_Δ converges to θ which is the unique continuous viscosity solution[16] of

[16] Uniqueness of continuous solutions can be shown using similar comparison principle arguments as in the above proof.

$$\begin{cases} -\gamma\partial_t\theta(t,q) + \frac{1}{2}\gamma^2\sigma^2q^2 - H\left(\partial_q\theta(t,q)\right) = 0, & \text{on } [0,T)\times(0,1] \\ \theta(t,q) = 0, & \text{on } [0,T]\times\{0\} \\ \theta(t,q) = -\ell(q)q, & \text{on } \{T\}\times[0,q_0] \end{cases}$$

with the limit condition and boundary condition satisfied in the classical sense.

Now, we are going to relate this equation to a classical equation in the Almgren-Chriss model of optimal liquidation. The intuition behind this link between our framework in the limit regime $\Delta \to 0$ and the Almgren-Chriss framework is that non-execution risk vanishes as Δ tends to 0. Hence, the only remaining risk is price risk, corresponding to the $\frac{1}{2}\gamma^2\sigma^2q^2$ term in the above equation. To see more precisely the correspondence between the two approaches, let us write the hamiltonian function as:

$$\begin{aligned} H(p) &= \gamma \sup_{\delta} \Lambda(\delta)(\delta - p) \\ &= \gamma \sup_{v>0} v(\Lambda^{-1}(v) - p) \\ &= \gamma \sup_{v\geq 0} v\Lambda^{-1}(v) - pv \end{aligned}$$

where the last equality holds since $\lim_{\delta\to+\infty} \delta\Lambda(\delta) = 0$.

Hence, if we define for $v > 0$, $f(v) = -\Lambda^{-1}(v)$, then we can define the hamiltonian function $\tilde{H}(p) = \sup_{v\geq 0} -f(v)v - pv$ and write the partial differential equation for θ as:

$$\begin{cases} -\partial_t\theta(t,q) + \frac{1}{2}\gamma\sigma^2q^2 - \tilde{H}\left(\partial_q\theta(t,q)\right) = 0, & \text{on } [0,T)\times(0,1] \\ \theta(t,q) = 0, & \text{on } [0,T]\times\{0\} \\ \theta(t,q) = -\ell(q)q, & \text{on } \{T\}\times[0,q_0] \end{cases}$$

This equation is the Hamilton-Jacobi equation associated to the Almgren-Chriss optimal liquidation problem with an instantaneous market impact function (per share) f and with a final discount $\ell(q_T)q_T$ at the time horizon[17] T. However, the instantaneous market impact function (sometimes also phrased execution cost function) has here a rather unusual form since $f(v)$ is negative for $v < \Lambda(0)$ and positive for $v > \Lambda(0)$, whereas it is usually a positive function. This must be interpreted in a very simple way: if one needs to obtain an instantaneous volume inferior to $\Lambda(0)$, then one will choose a positive δ, that is he will send a limit order[18] – this makes sense since non-execution risk disappears in the limit regime $\Delta \to 0$; on the contrary, if one needs an instantaneous volume greater than $\Lambda(0)$, then he will rely on a market order ($\delta < 0$).

The above discussion only makes sense at the limit, when non-execution risk does not exist anymore. However, it provides some intuition about the way to understand the model, especially when it comes to negative δ. In particular, we will see that the above correspondence between our model and a model *à la* Almgren-Chriss provides a possible way to solve one of the main problems of the model discussed in [25]: the interpretation of the exponential intensity functions for negative values of δ.

[17] In the usual Almgren-Chriss framework, this discount is in fact $-\infty 1_{q_T>0}$ to force liquidation but the theory can easily be adapted to incorporate a finite discount.

[18] since we assumed that the reference price is the first bid quote.

5.2 Application to the choice of Λ

The model we discuss in this paper does not consider explicitly market order or limit orders but rather considers that there is, for each price $s^a = s + \delta$, an instantaneous probability to obtain a trade at that price. In practice, this interpretation is well suited for limit orders, that is for positive values of δ, but we need to provide an interpretation for negative values of δ and similarly provide an intuition for the signification of the intensity function Λ on the entire real line. In practice, since the model has been designed to liquidate a position with limit orders, it should not be used if the liquidation under scrutiny evidently requires liquidity-consuming orders. However, it may happen, because of a slow execution, that the optimal quote in the model turns out to be negative[19] after some time.

This issue of negative δ was present in our initial model with exponential intensity functions (see [25]). Although the exponential form was justified for some stocks when $\delta \geq 0$, the intensity function was also exponential for $\delta < 0$ and this choice was rather dictated by mathematical needs than by empirical rationale. Now, since Λ (or in practice Λ_Δ) can be chosen, we can improve by far the initial model.
First, using statistics on execution, we can estimate the probability to be executed at any positive distance of the first bid limit. In practice the profile of the empirical intensity for positive δ is decreasing and may not be convex, especially when the bid-ask spread is large. This is an important remark because in the case of a non-convex Λ_Δ the hamiltonian function H_Δ has no reason to be convex either and then, the equation ($\mathrm{HJ}_{\theta_\Delta}$) is not directly associated to a deterministic control problem.

Then, once the function Λ_Δ has been calibrated for positive δ, several natural choices are possible for $\Lambda_\Delta(\delta)$ when $\delta \leq 0$. Instead of extending the function for negative δ using a specific functional form as in [25], we can assign to $\Lambda_\Delta(\delta)$ a constant value when $\delta \leq 0$, this constant being equal to the value of $\Lambda_\Delta(0^+)$[20]. This corresponds to a very conservative choice that basically prevents the use of market order since the optimal quote will always be positive. However, a better choice consists in using the parallel made between the usual literature and our framework when the limit regime $\Delta \to 0$ is considered. If we indeed omit the non-execution risk, we can consider, for $\delta \leq 0$, that $\Lambda_\Delta(\delta) = \frac{1}{\Delta} f^{-1}(-\delta) = \frac{1}{\Delta} \sup\{v \geq 0, f(v) \leq -\delta\}$ where $v \mapsto f(v)$ is an instantaneous market impact function (average execution cost per share) that is typically equal to nought for small values of v and increasing.

This choice for Λ_Δ is however subject to several comments. First, the function Λ_Δ must satisfy the hypotheses of the model. In particular, it must be decreasing and it may happen that, although the specifications for positive δ and negative δ are decreasing, the function is not decreasing on the entire real line. Since the function $\delta \leq 0 \mapsto \frac{1}{\Delta} f^{-1}(-\delta)$ is rather a lower bound to $\Lambda_\Delta(\delta)$ because we have to take account of the non-execution risk, we can always scale the function $\delta \leq 0 \mapsto \Lambda_\Delta(\delta)$ so that the resulting function Λ_Δ is decreasing (and strictly decreasing if we smooth the function). Second, a question remains regarding the interpretation of the model when an optimal quote δ^* turns out to be negative. The answer we propose, in line with the parallel made with Almgren-Chriss-like models, is to send a market order of size $\Lambda(\delta^*) = \Delta\Lambda_\Delta(\delta^*)$.

[19] Although quotes evolve continuously between execution times, the practical use of the model (see [25]) requires to stay in the limit order book for some time and the optimal quote at some point may in practice be a strictly negative figure.

[20] Or in fact $\Lambda_\Delta(\textit{tick size})$.

Conclusion

In this paper, we analyze the optimal liquidation strategy of a trader with a single-stock portfolio using limit orders. The classical literature on optimal liquidation, following Almgren-Chriss, only answers the question of the optimal liquidation rhythm and a new literature has recently emerged that uses either dark pools or limit orders to tackle the issue of the actual optimal way to liquidate. Our paper provides a general model for optimal liquidation with limit orders and extends both [13] that only considers a risk-neutral framework and [25] that was restricted to exponential intensity functions. Our results can be extended easily with the adjunction of a drift term[21] and more importantly to the case of a multi-stock portfolio, taking into account correlation between stock prices. Another improvement would also consist in linking the Brownian motion which drives the price and the point process modeling execution. Research in this direction has recently been made by Cartea, Jaimungal and Ricci [18] to model market making and a simplified version of their approach could be introduced in our model. For practical purposes, the most important improvement would consist in introducing the bid-ask spread of the market in order to better account for changes in the probability to be executed at a given price. All these improvements are part of an ongoing research work.

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